NOTES ON CATEGORY THEORY

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Abstract. These notes serve as a compilation of foundational material in category theory, intended primarily as a personal reference to help understand recent developments in representation theory. While no original results are presented, this text provides a self-contained exposition of three central themes: abelian categories, tensor categories, and stratification of abelian categories.

The material is organized into four distinct parts. Part I establishes the theory of abelian categories, building up from the axioms of abstract categories and universal constructions (including limits, colimits, kernels, and cokernels) through pre-additive and additive structures. Part II acts as a bridge, detailing some types of functors, such as exact, fully faithful, and adjoint functors, that are essential for the remainder of the text. Part III focuses on categories equipped with internal multiplication, systematically defining and exemplifying monoidal, braided, and symmetric monoidal categories. Finally, Part IV addresses the stratifications of abelian categories. In this section, we introduce the machinery of Serre subcategories and Serre quotients, culminating in the theory of recollements and stratifications.

Contents

Part I

Abelian Categories

The main goal of this part is to introduce *abelian categories.* This concept distils the essential properties of abelian groups and provides an appropriate axiomatic framework to study several objects in algebra, topology and algebraic geometry.

We will build the theory of abelian categories constructively, moving from the general to the specific. We begin with the fundamental definitions of abstract categories and basic constructions. Subsequently, we enrich this structure by introducing preadditive and additive categories, which impose an abelian group structure on the sets of morphisms. Finally, by necessitating the existence of kernels and cokernels, we arrive at the definition of pre-abelian and, ultimately, abelian categories. On the route through these concepts, we provide several illustrative examples.

1.1. Abstract categories

We will begin by presenting the abstract definition of a category. Categories were created to provide a language that unifies different mathematical fields, enabling the transfer of results and ideas between them. This universality comes with a lot of flexibility, but one immediately notices that it also comes with a lot of abstraction.

Definition 1.1.1. A *category* $\mathcal{C}$ is a triple $(\mathrm{Obj}, \mathrm{Mor}, \circ)$, where:

- Obj is a class, whose elements are called the *objects* of $\mathcal{C}$;
- Mor is also a class, whose elements are called the *morphisms* of $\mathcal{C}$, and moreover, for each pair of objects A and B in Obj, there is a subclass of Mor, denoted by $\mathrm{Hom}(A, B)$, whose elements are called morphisms from A to B and often denoted by arrows $A \to B$;
- $\circ$ is a relation $\mathrm{Mor} \times \mathrm{Mor} \to \mathrm{Mor}$ called *composition*, that satisfies the following conditions:

(i) for every three objects A, B, C and two morphisms $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, there exists a morphism $(g \circ f) \in \mathrm{Hom}(A, C)$.

(ii) for every four objects A, B, C, D in Obj and every three morphisms $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(C, D)$, we have:

$$(h \circ g) \circ f = h \circ (g \circ f);$$

(iii) for every object A in Obj, there is a morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ called the *identity morphism*, that satisfies the following conditions: for every morphism $f \in \mathrm{Hom}(A, B)$, we have $f \circ \mathrm{id}_A = f$, and for every morphism $g \in \mathrm{Hom}(B, A)$, we have $g \circ \mathrm{id}_B = g$.

To help make this abstract definition more concrete, we will now provide two examples of categories. We begin with the smallest possible category for which Obj is non-empty.

Example 1.1.2. The smallest category for which Obj is non-empty is the one with only one object and one morphism. More explicitly:

- $\mathrm{Obj} = \{\star\}$, where $\star$ is the only object of this category;
- $\mathrm{Mor} = \{\mathrm{id}_\star\}$, that is, $\mathrm{id}_\star$ is the only morphism of this category;
- the composition is given by $\mathrm{id}_\star \circ \mathrm{id}_\star = \mathrm{id}_\star$.

Notice that, in fact, all the conditions given in Definition 1.1.1 hold.

The next example is a more concrete and well-known one. It shows how sets and functions can form a category.

Example 1.1.3. The category of sets, which we will denote by **Sets**, is given as follows:

- The objects of the category **Sets** are all the sets;
- The morphisms of **Sets** are all the functions between sets;
- The composition of morphisms is the usual composition of functions.

Notice that all the conditions given in Definition 1.1.1 hold. In fact:

(i) for every three sets A, B, C and two functions $f : A \to B$, $g : B \to C$, their composition $(g \circ f) : A \to C$ is also a function.

(ii) for every four sets A, B, C, D and every three functions $f : A \to B$, $g : B \to C$, $h : C \to D$, we have:

$$\begin{aligned}((h \circ g) \circ f)(a) &= (h \circ g)(f(a)) \\ &= h(g(f(a))) \\ &= h((g \circ f)(a)) \\ &= (h \circ (g \circ f))(a), \qquad \text{for all } a \in A.\end{aligned}$$

(iii) for every set A, its identity function is the function $\mathrm{id}_A : A \to A$ explicitly given by $\mathrm{id}_A(a) = a$ for all $a \in A$. Notice that, for every set B, every function $f : A \to B$ and every function $g : B \to A$, we have:

$$\begin{aligned}(f \circ \mathrm{id}_A)(a) = f(\mathrm{id}_A(a)) = f(a) \quad \text{for all } a \in A, \\ (\mathrm{id}_A \circ g)(b) = \mathrm{id}_A(g(b)) = g(b) \quad \text{for all } b \in B.\end{aligned}$$

We will see further examples of categories in the following sections. To finish this section, we would like to make a few technical remarks.

Notice that in the definition of category, objects and morphisms form *classes* rather than *sets*. This distinction arises because, in set theory, one faces limitations when trying to define collections that are "too large" or "too general". In fact, in standard set theory (such as Zermelo-Fraenkel set theory with the Axiom of Choice, or ZFC) a set is a collection of elements that is itself an element of some larger set. However, there are strict limitations on the size of a set due to the *Axiom of Regularity* (which prevents sets from containing themselves directly or indirectly) and the *Axiom of Infinity* (which ensures that no set is "too large").

In particular, the collection of all sets cannot itself be a set because such a collection would lead to paradoxes like Russell's paradox, which questions whether a set of all sets that do not contain themselves contains itself. To avoid such paradoxes, one introduces the notion of classes for collections that may be "too large" to be sets, but are still useful in formalising mathematical concepts. In particular, this allows the category **Sets** from Example 1.1.3 to be a category.

However, as we have seen in Example 1.1.2, there are instances where the objects or morphisms of a category form a set. In the cases where the morphisms form a set, the category is called *locally small*, and in the cases where the objects also form a set, the category is called *small*. Most of the categories that we will deal with in the coming sections will be either locally small or small.

1.2. Constructions in categories I

In this section, we will present some constructions in abstract categories. At first sight, some of these constructions can seem too abstract. In the examples, we provide more concrete instances of these abstract definitions, and in subsequent sections, we will use them.

1.2.1. Isomorphisms.

In this subsection, we will introduce isomorphisms, which are central to understanding how objects in a category relate to each other. More specifically, two objects are said to be isomorphic when they are essentially the same from the perspective of the category. This relationship is formally captured by the following definition.

Definition 1.2.1. Given a category $\mathcal{C}$ and two objects $A, B \in \mathrm{Obj}$, a morphism $f \in \mathrm{Hom}(A, B)$ is said to be an *isomorphism* when there exists a morphism $g \in \mathrm{Hom}(B, A)$ such that:

$$g \circ f = \mathrm{id}_A \quad \text{and} \quad f \circ g = \mathrm{id}_B\,.$$

In this case, g is called the *inverse* of f and the objects A and B are said to be *isomorphic*.

This abstract definition of isomorphism captures the idea of sameness between elements. In the following examples, we will see how this abstract definition works in more concrete cases.

Example 1.2.2. Let $\mathcal{C}$ be a category such that Obj is non-empty. For every object $X \in \mathrm{Obj}$, its identity morphisms is an isomorphism. In fact, recall that $\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X$. This implies that id_X is the inverse of itself, that id_X is an isomorphism, and that X is isomorphic to itself.

In the first example above, we considered identities as isomorphisms. However, isomorphisms can be more general and relate distinct objects. In the next example we will see how isomorphisms are precisely the bijections in the category of sets.

Example 1.2.3. In the category of sets (see Example 1.1.3), isomorphisms are bijections. To justify that, we begin by recalling that in the category of sets, the objects are sets, and the morphisms are functions between sets. In particular, a morphism (function) f from an object (set) A to an object (set) B is an isomorphism when there exists a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. This means that f and g are bijections between A and B.

1.2.2. **Initial and terminal objects.** In this subsection, we turn our attention to the concepts of initial and terminal objects. Their definition is the first one in which a "universal property" appears.

Definition 1.2.4. An object I in a category $\mathcal{C}$ is said to be an *initial object* if, for every object X in Obj, there is a unique morphism from I to X. Similarly, an object T in $\mathcal{C}$ is said to be a *terminal object* if, for every object X in Obj, there is a unique morphism from X to T.

To see how these definitions are realized in concrete cases, we will exhibit the initial and terminal objects in two categories that we have already introduced. We begin by showing how the unique object of the smallest category for which Obj is non-empty is both initial and terminal.

Example 1.2.5. In the category constructed in Example 1.1.2, the unique object $\star$ is both an initial and a terminal object. In fact, since $\star$ is the only object in this category, the identity morphism satisfies $\mathrm{id}_\star : \star \to \star$, and $\mathrm{id}_\star$ is the only morphism in this category, $\star$ satisfies the conditions for being both an initial and a terminal object in this category.

The category from Example 1.2.5 is particularly simple, as it has only one object, and thus both initial and terminal object conditions are trivially satisfied. To provide a more substantial example, we now consider the category of sets, where the concepts of initial and terminal objects are a little less trivial.

Example 1.2.6. In the category of sets (see Example 1.1.3), the empty set is the only initial object, and every set with one element is a terminal object.

To justify that the empty set, $\emptyset$, is an initial object in the category **Sets**, we must show that, for every set X, there exists a unique function $f : \emptyset \to X$. Since $\emptyset$ has no elements, there are no elements to be mapped, and thus, the empty function is the only function from the $\emptyset$ to X.

Now, to justify that a set $\{\star\}$ with one element ($\star$) is terminal in **Sets**, we need to show that, for every set X, there exists exactly one function from X to $\{\star\}$. Since $\{\star\}$ has only one element, any function $f : X \to \{\star\}$ must assign every element $x \in X$ to $\star$. Thus, for every set X, there exists only the constant function from X to $\{\star\}$.

The examples above highlight the general idea that initial objects "map to" every other object uniquely, while terminal objects "receive a unique mapping" from every other object. The next proposition formalizes a key property of initial and terminal objects, namely, that when they exist, initial and terminal objects are essentially (up to isomorphism) unique in a category.

Proposition 1.2.7. Let $\mathcal{C}$ be a category.

(a) If I and I' are initial objects of $\mathcal{C}$, then I and I' are isomorphic.
(b) If T and T' are terminal objects of $\mathcal{C}$, then T and T' are isomorphic.

Proof. We will prove each part of this proposition separately.

(a) Suppose that I and I' are both initial objects in $\mathcal{C}$. We want to prove that I and I' are isomorphic.

By the definition of an initial object, for each object X in $\mathcal{C}$, there is a unique morphism from I to X, and a unique morphism from I' to X. In particular, there exists a unique morphism from I to I', say $f : I \to I'$, and also a unique morphism from I' to I, say $g : I' \to I$.

Now, consider the composition $g \circ f : I \to I$. Since I is initial, there is a unique morphism from I to itself, which must be the identity morphism id_I. Therefore, we have:

$$g \circ f = \mathrm{id}_I .$$

Similarly, consider the composition $f \circ g : I' \to I'$. Since I' is initial, there is a unique morphism from I' to itself, which must be the identity morphism $\mathrm{id}_{I'}$. Therefore, we have:

$$f \circ g = \mathrm{id}_{I'} .$$

Thus, f and g are mutually inverse, so f is an isomorphism, and as a consequence I is isomorphic to I'.

(b) Suppose now that T and T' are both terminal objects in $\mathcal{C}$. We want to prove that T and T' are isomorphic.

By the definition of a terminal object, for each object X in $\mathcal{C}$, there is a unique morphism from X to T, and a unique morphism from X to T'. In particular, there exists a unique morphism from T to T', say $f : T \to T'$, and also a unique morphism from T' to T, say $g : T' \to T$.

Now, consider the composition $f \circ g : T' \to T'$. Since T' is terminal, there is a unique morphism from T' to itself, which must be the identity morphism $\mathrm{id}_{T'}$. Therefore, we have:

$$f \circ g = \mathrm{id}_{T'}\,.$$

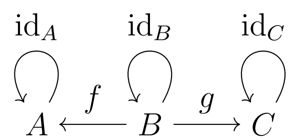
Similarly, consider the composition $g \circ f : T \to T$. Since T is terminal, there is a unique morphism from T to itself, which must be the identity morphism id_T. Therefore, we have:

$$g \circ f = \mathrm{id}_T\,.$$

Thus, f and g are mutually inverse, so f is an isomorphism, and as a consequence, T is isomorphic to T'. □

It is important to note that not all categories have an initial or terminal object. The existence of these objects depends on the specific structure of the category in question. We end this subsection showing an example of a category with an initial object and without any terminal one.

Example 1.2.8. Consider a category $\mathcal{C}$ with three objects, $\mathrm{Obj} = \{A, B, C\}$, and five morphisms, $\mathrm{Mor} = \{\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C, f, g\}$, where $f \in \mathrm{Hom}(B, A)$ and $g \in \mathrm{Hom}(B, C)$. A diagrammatic picture of this category is the following:



In this case, the object B is an initial object, but no object in this category is terminal. To justify the claim that B is an initial object, notice that

$$\mathrm{Hom}(B, A) = \{f\}, \quad \mathrm{Hom}(B, B) = \{\mathrm{id}_B\} \quad \text{and} \quad \mathrm{Hom}(B, C) = \{g\},$$

that is, there exists exactly one morphism from B to every object in $\mathcal{C}$. Now, to justify the claim that no object in $\mathcal{C}$ is terminal, notice that

$$\mathrm{Hom}(C, A) = \emptyset, \quad \mathrm{Hom}(A, B) = \mathrm{Hom}(C, B) = \emptyset \quad \text{and} \quad \mathrm{Hom}(A, C) = \emptyset.$$

In a similar way as in the example above, one can construct a category with no initial object and a terminal object, or a category with no initial object and no terminal object, or a category with several isomorphic initial (or terminal) objects.

1.2.3. **Products.** Products are the abstract categorical concept that generalizes Cartesian products and direct products. They provide a way to combine objects in a category into a single object that "projects" back to the original objects in a universal way. In this section, we will define products abstractly and explore concrete examples to illustrate their properties.

Definition 1.2.9. Given a category $\mathcal{C}$, the *product* of two of its objects, A and B, is a triple (P, p_1, p_2) that satisfies the following properties:

(i) $P \in \mathrm{Obj}$,

(ii) $p_1 \in \mathrm{Hom}(P, A)$,

(iii) $p_2 \in \mathrm{Hom}(P, B)$,

(iv) for every object X in Obj and every pair of morphisms $f_1 \in \mathrm{Hom}(X, A)$ and $f_2 \in \mathrm{Hom}(X, B)$, there exists a unique morphism $F : X \to P$ such that $p_1 \circ F = f_1$ and $p_2 \circ F = f_2$.

To better understand the abstract definition of products, we will consider some concrete examples. We will start with the simplest possible category and then move to more familiar categories like the category of sets.

Example 1.2.10. Let $\mathcal{C}$ be a category with only one object and only one morphism, defined in Example 1.1.2. The product of its only object $\star$ with itself is the triple $(\star, \mathrm{id}_\star, \mathrm{id}_\star)$.

To justify this, notice that $\star \in \mathrm{Obj}$ and $\mathrm{id}_\star \in \mathrm{Hom}(\star, \star)$. Also notice that, since this category only has the object $\star$ and the morphism $\mathrm{id}_\star$, then we only need to analyse the case where $X = \star$ and $f_1 = f_2 = \mathrm{id}_\star : \star \to \star$. In fact, there exists the morphism $F = \mathrm{id}_\star : \star \to \star$ that satisfies $\mathrm{id}_\star \circ \mathrm{id}_\star = \mathrm{id}_\star$.

Having seen the simplest example of a product, we now consider a more familiar category: the category of sets. Here, products correspond to the Cartesian product of sets.

Example 1.2.11. In the category of sets, the product of two non-empty sets is the Cartesian product of these sets equipped with their respective projections. To justify this, let A and B be sets, recall that their Cartesian product is the set defined by

$$A \times B := \{(a, b) \mid a \in A,\ b \in B\},$$

and define the functions

$$p_1 : (A \times B) \to A \quad \text{given by} \quad p_1(a, b) = a,$$
$$p_2 : (A \times B) \to B \quad \text{given by} \quad p_2(a, b) = b.$$

Given a set X and two functions, $f_1 : X \to A$ and $f_2 : X \to B$, define a function $F : X \to (A \times B)$ as being given by $F(x) = (f_1(x), f_2(x))$. Notice that F is a well-defined function, that $(p_1 \circ F)(x) = p_1(F(x)) = f_1(x)$ and that $(p_2 \circ F)(x) = p_2(F(x)) = f_2(x)$ for all $x \in X$. This justifies that the triple $(A \times B, p_1, p_2)$ is, in fact, the product of the objects A and B in the category of sets.

The following proposition formalizes the fact that products are unique up to isomorphism, meaning that any two products of the same pair of objects are essentially the same in a categorical sense.

Proposition 1.2.12. Let $\mathcal{C}$ be a category, and let A and B be objects of $\mathcal{C}$. If (P, p_1, p_2) and (P', p_1', p_2') are two products of A and B, then P is isomorphic to P'.

Proof. First, recall from the construction of (P, p_1, p_2) that $p_1 \in \mathrm{Hom}(P, A)$ and $p_2 \in \mathrm{Hom}(P, B)$. Then, recall from the defining property of (P', p_1', p_2'), that there exists a unique morphism $f : P \to P'$ such that $p_1' \circ f = p_1$ and $p_2' \circ f = p_2$.

Similarly, by switching P and P', we see that there exists a unique morphism $f' : P' \to P$ such that $p_1 \circ f' = p_1'$ and $p_2 \circ f' = p_2'$. Hence, $(f' \circ f) : P \to P$ is a morphism such that:

$$p_1 \circ (f' \circ f) = (p_1 \circ f') \circ f = p_1' \circ f = p_1$$

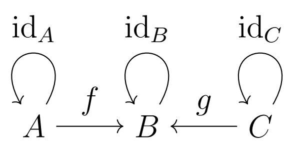
and

$$p_2 \circ (f' \circ f) = (p_2 \circ f') \circ f = p_2' \circ f = p_2.$$

Further, recall from the construction of (P, p_1, p_2) that $p_1 \in \mathrm{Hom}(P, A)$ and $p_2 \in \mathrm{Hom}(P, B)$. Then, recall from the defining property of (P, p_1, p_2) itself, that there exists a unique morphism $\phi : P \to P$ such that $p_1 \circ \phi = p_1$ and $p_2 \circ \phi = p_2$. Since $\phi = \mathrm{id}_P$ satisfies these conditions and, as we have shown above, $\phi = f' \circ f$ also satisfies these conditions, we conclude that $f' \circ f = \mathrm{id}_P$. Thus, $f : P \to P'$ and $f' : P' \to P$ are isomorphisms. $\square$

In the proposition above, we showed that products are unique, when they exist. However, not all categories have products for every pair of objects. In the next example, we will see a simple category where the product of two objects does not exist.

Example 1.2.13. Consider a category $\mathcal{C}$ with three objects, $\mathrm{Obj} = \{A, B, C\}$, and five morphisms, $\mathrm{Mor} = \{\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C, f, g\}$, where $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(C, B)$. A diagrammatic picture of this category is the following:



In this case, the product of the objects A and C does not exist. In fact, there exists no object P in $\mathcal{C}$ such that $\mathrm{Hom}(P, A)$ and $\mathrm{Hom}(P, C)$ are simultaneously non-empty:

$$\mathrm{Hom}(A, C) = \emptyset, \quad \mathrm{Hom}(B, A) = \mathrm{Hom}(B, C) = \emptyset \quad \text{and} \quad \mathrm{Hom}(C, A) = \emptyset.$$

Notice that in Definition 1.2.9, we have defined the product of two objects in an abstract category. However, one can define, in a similar way, the product of any family of objects. Specifically, if $\{A_i\}_{i \in I}$ is a family of objects in a category $\mathcal{C}$ (I being its indexing set), then its product is an object $P \in \mathrm{Obj}$ together with a family of morphisms $\{p_i\}_{i \in I}$ satisfying the following properties:

- $p_i : P \to A_i$ for each $i \in I$,
- for any object $X \in \mathrm{Obj}$ for which a family of morphisms $\{f_i : X \to A_i\}_{i \in I}$ exists, there exists also a unique morphism $F : X \to P$ such that $p_i \circ F = f_i$ for all $i \in I$.

In particular, in the case where the index set I is empty, we have the following result:

Proposition 1.2.14. Let $\mathcal{C}$ be a category. If the product of an empty family of objects in $\mathcal{C}$ exists, then it is a terminal object of $\mathcal{C}$.

Proof. By definition, the empty product is an object P such that for any object X, there exists a unique morphism $f : X \to P$. Therefore, if the empty product exists, it must be a terminal object of $\mathcal{C}$. $\square$

1.2.4. **Coproducts.** Coproducts are the abstract categorical concept that generalizes disjoint unions. They provide a way to combine objects in a category into a single object that "contains a copy" of the original objects in a universal way. In this section, we will define coproducts abstractly and explore concrete examples to illustrate their properties.

Definition 1.2.15. Given a category $\mathcal{C}$, the *coproduct* of two of its objects, A and B, is a triple (C, i_1, i_2) that satisfy the following properties:

(i) $C \in \mathrm{Obj}$,

(ii) $i_1 \in \mathrm{Hom}(A, C)$,

(iii) $i_2 \in \mathrm{Hom}(B, C)$,

(iv) for every object X in Obj and every pair of morphisms $f_1 \in \mathrm{Hom}(A, X)$ and $f_2 \in \mathrm{Hom}(B, X)$, there exists a unique morphism $F : C \to X$ such that $F \circ i_1 = f_1$ and $F \circ i_2 = f_2$.

To better understand the abstract definition of coproducts, we will consider some concrete examples. We will start with the simplest possible category and then move to more familiar categories like the category of sets.

Example 1.2.16. Let $\mathcal{C}$ be a category with only one object and only one morphism, defined in Example 1.1.2. The coproduct of its only object $\star$ with itself is the triple $(\star, \mathrm{id}_\star, \mathrm{id}_\star)$.

To justify this, notice that $\star \in \mathrm{Obj}$ and $\mathrm{id}_\star \in \mathrm{Hom}(\star, \star)$. Also notice that, since this category only has the object $\star$ and the morphism $\mathrm{id}_\star$, then we only need to analyse the case where $X = \star$ and $f_1 = f_2 = \mathrm{id}_\star : \star \to \star$. In fact, there exists the morphism $F = \mathrm{id}_\star : \star \to \star$ that satisfies $\mathrm{id}_\star \circ \mathrm{id}_\star = \mathrm{id}_\star$.

Having seen the simplest example of a coproduct, we now consider a more familiar category: the category of sets. Here, products correspond to the disjoint union of sets.

Example 1.2.17. In the category of sets, the coproduct of two non-empty sets is the disjoint union of these sets equipped with their respective inclusions. To justify this, let A and B be sets, recall that their disjoint union is the set defined by

$$A \sqcup B := \{x_a \mid a \in A\} \cup \{y_b \mid b \in B\},$$

where we treat x_a and y_b as formal elements in order to differentiate elements that may eventually be in the intersection $A \cap B$. Then, define the inclusions

$$i_1 : A \to (A \sqcup B) \quad \text{given by} \quad i_1(a) = x_a,$$
$$i_2 : B \to (A \sqcup B) \quad \text{given by} \quad i_2(b) = y_b.$$

Given a set X and two functions, $f_1 : X \to A$ and $f_2 : X \to B$, define a function $F : (A \sqcup B) \to X$ as being given by

$$F(z) = \begin{cases} f_1(a), & \text{if } z = x_a \text{ for some } a \in A, \\ f_2(b), & \text{if } z = y_b \text{ for some } b \in B. \end{cases}$$

Notice that F is well-defined, that $(F \circ i_1)(a) = F(i_1(a)) = f_1(x_a) = a$ for all $a \in A$ and that $(F \circ i_2)(b) = F(i_2(b)) = f_2(y_b) = b$ for all $b \in B$. This justifies that the triple $(A \sqcup B, i_1, i_2)$ is, in fact, the coproduct of the objects A and B in the category of sets.

The following proposition formalizes the fact that coproducts are also unique up to isomorphism, meaning that any two coproducts of the same pair of objects are essentially the same in a categorical sense.

Proposition 1.2.18. Let $\mathcal{C}$ be a category, and let A and B be objects of $\mathcal{C}$. If (C, i_1, i_2) and (C', i'_1, i'_2) are two coproducts of A and B, then C is isomorphic to C'.

Proof. First, recall from the construction of (C, i_1, i_2) that $i_1 \in \mathrm{Hom}(A, C)$ and $i_2 \in \mathrm{Hom}(B, C)$. Then, recall from the defining property of (C', i'_1, i'_2), that there exists a unique morphism $I' : C \to C'$ such that $I' \circ i_1 = i'_1$ and $I' \circ i_2 = i'_2$.

Similarly, by switching C and C', we see that there exists a unique morphism $I : C' \to C$ such that $I \circ i'_1 = i_1$ and $I \circ i'_2 = i_2$. Hence, $(I \circ I') : C \to C$ is a morphism such that:

$$(I \circ I') \circ i_1 = I \circ (I' \circ i_1) = I \circ i'_1 = i_1$$

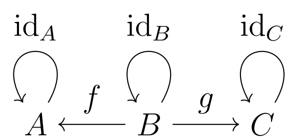
and

$$(I \circ I') \circ i_2 = I \circ (I' \circ i_2) = I \circ i'_2 = i_2.$$

Further, recall from the construction of (C, i_1, i_2) that $i_1 \in \mathrm{Hom}(A, C)$ and $i_2 \in \mathrm{Hom}(B, C)$. Then, recall from the defining property of (C, i_1, i_2) itself, that there exists a unique morphism $\phi : C \to C$ such that $\phi \circ i_1 = i_1$ and $\phi \circ i_2 = i_2$. Since $\phi = \mathrm{id}_C$ satisfies these conditions and, as we have shown above, $\phi = I \circ I'$ also satisfies these conditions, we conclude that $I \circ I' = \mathrm{id}_C$. Thus, $I' : C \to C'$ and $I : C' \to C$ are isomorphisms. ☐

In the proposition above, we showed that coproducts are unique, when they exist. However, not all categories have coproducts for every pair of objects. In the next example, we will see a simple category where the coproduct of two objects does not exist.

Example 1.2.19. Consider a category $\mathcal{C}$ with three objects, $\mathrm{Obj} = \{A, B, C\}$, and five morphisms, $\mathrm{Mor} = \{\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C, f, g\}$, where $f \in \mathrm{Hom}(B, A)$ and $g \in \mathrm{Hom}(B, C)$. A diagrammatic picture of this category is the following:



In this case, the coproduct of the objects A and C does not exist. In fact, there exists no object X in $\mathcal{C}$ such that $\mathrm{Hom}(A, X)$ and $\mathrm{Hom}(C, X)$ are simultaneously non-empty:

$$\mathrm{Hom}(C, A) = \emptyset, \quad \mathrm{Hom}(A, B) = \mathrm{Hom}(C, B) = \emptyset \quad \text{and} \quad \mathrm{Hom}(A, C) = \emptyset.$$

Notice that in Definition 1.2.15 we have defined the coproduct of two objects in an abstract category. However, one can define, in a similar way, the coproduct of any family of objects. Specifically, if $\{A_j\}_{j \in J}$ is a family of objects in a category $\mathcal{C}$ (J being its indexing set), then its coproduct is an object $C \in \mathrm{Obj}$, together with a family of morphisms $\{\iota_j\}_{j \in J}$ satisfying the following properties:

- $\iota_j : A_j \to C$ for each $j \in J$,
- For any object $X \in \mathrm{Obj}$ for which a family of morphisms $\{f_j : A_j \to X\}_{j \in J}$ exists, there exists also a unique morphism $F : C \to X$ such that $F \circ \iota_j = f_j$ for all $j \in J$.

In particular, in the case where the index set J is empty, we have the following result:

Proposition 1.2.20. Let $\mathcal{C}$ be a category. If the coproduct of an empty family of objects in $\mathcal{C}$ exists, then it is an initial object of $\mathcal{C}$.

Proof. By definition, the empty coproduct is an object C such that for any object X, there exists a unique morphism $F : C \to X$. Therefore, if the empty coproduct exists, it must be an initial object of $\mathcal{C}$. □

1.3. Preadditive categories

Preadditive categories are categories for which the hom-sets are equipped with an abelian group structure (see Definition A.1). This structure allows for the addition of morphisms. In this section, we will define preadditive categories and explore their properties through concrete examples and propositions.

Definition 1.3.1. A category $\mathcal{C}$ is said to be *preadditive* when, for every pair of objects $A, B \in \mathrm{Obj}$, there exists a binary operation

$$+ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B),$$

such that $(\mathrm{Hom}(A, B), +)$ is an abelian group and, moreover,

$$(f + f') \circ g = f \circ g + f' \circ g \quad \text{ and } \quad g \circ (f + f') = g \circ f + g \circ f',$$

for all objects $A, B, C, D \in \mathrm{Obj}$, all morphisms $f, f' \in \mathrm{Hom}(A, B)$, and all morphisms $g, g' \in \mathrm{Hom}(C, D)$.

Notice that if a category $\mathcal{C}$ is preadditive, then it is also locally small, since $\mathrm{Hom}(A, B)$ must be a set for every pair of objects $A, B \in \mathrm{Obj}$. Notice, moreover, that if $\mathcal{C}$ is a preadditive category, then $\mathrm{Hom}(A, B)$ is non-empty for every pair of objects $A, B \in \mathrm{Obj}$.

To better understand the abstract definition of preadditive categories, we will consider some concrete examples. We will start with the simplest possible category and then explore more complex cases.

Example 1.3.2. Notice that the category with one object and one morphism (constructed in Example 1.1.2) is preadditive. In fact, recall that this category has only one object, $\mathrm{Obj} = \{\star\}$ and only one morphism, $\mathrm{Mor} = \{\mathrm{id}_\star\}$. Hence, in this case, the only group structure that the set Mor (with only one element) admits is the trivial one, that is, $\mathrm{id}_\star + \mathrm{id}_\star := \mathrm{id}_\star$. Thus, it is obvious that:

$$\mathrm{id}_\star \circ (\mathrm{id}_\star + \mathrm{id}_\star) = \mathrm{id}_\star \circ \mathrm{id}_\star = \mathrm{id}_\star = \mathrm{id}_\star + \mathrm{id}_\star = (\mathrm{id}_\star \circ \mathrm{id}_\star) + (\mathrm{id}_\star \circ \mathrm{id}_\star),$$
$$(\mathrm{id}_\star + \mathrm{id}_\star) \circ \mathrm{id}_\star = \mathrm{id}_\star \circ \mathrm{id}_\star = \mathrm{id}_\star = \mathrm{id}_\star + \mathrm{id}_\star = (\mathrm{id}_\star \circ \mathrm{id}_\star) + (\mathrm{id}_\star \circ \mathrm{id}_\star).$$

This shows that the category constructed in Example 1.1.2 is, in fact, preadditive.

While the simplest category is preadditive, not all categories share this property. In the next example, we will see that the category of sets is not preadditive.

Example 1.3.3. Notice that the category of sets (see Example 1.1.3) is not preadditive, as it is not locally small.

Similarly, other categories may fail to be preadditive due to the lack of appropriate structure on their hom-sets. In the next example, we will see two such categories.

Example 1.3.4. Notice that the category constructed in Example 1.2.8 and the category constructed in Example 1.2.13 are not preadditive. In fact, in the first case, $\mathrm{Hom}(A, B) = \mathrm{Hom}(C, B) = \emptyset$ are not groups, and in the second case, $\mathrm{Hom}(B, A) = \mathrm{Hom}(B, C) = \emptyset$ are not groups.

Preadditive categories have several important properties that distinguish them from general categories. The following proposition highlights some of these properties, including the relationship between initial and terminal objects and the duality between products and coproducts.

Proposition 1.3.5. Let $\mathcal{C}$ be a preadditive category.

(a) An object of $\mathcal{C}$ is initial if and only if it is terminal.

(b) Let n be a non-negative integer and $\{A_1, \ldots, A_n\}$ be a finite subset of Obj. If the product of $A_1, \ldots, A_n$ exists, then their coproduct also exists. Moreover, in this case, the product of $A_1, \ldots, A_n$ is isomorphic to their coproduct.

(c) Let n be a non-negative integer and $\{A_1, \ldots, A_n\}$ be a finite subset of Obj. If the coproduct of $A_1, \ldots, A_n$ exists, then their product also exists. Moreover, in this case, the coproduct of $A_1, \ldots, A_n$ is isomorphic to their product.

Proof. We will prove each part of this proposition separately.

(a) Suppose I is an initial object in the category $\mathcal{C}$. We want to prove that I is also terminal, that is, we want to show that, for every object X in $\mathcal{C}$, the set $\mathrm{Hom}(X, I)$ contains exactly one morphism.

Let X be an object of $\mathcal{C}$. Since $\mathcal{C}$ is assumed to be a preadditive category, we know that $\mathrm{Hom}(X, I)$ is non-empty and, moreover, forms a group. Therefore, showing that $\mathrm{Hom}(X,I)$ contains exactly one element is equivalent to showing that the only morphism in $\mathrm{Hom}(X,I)$ is the neutral element of this group.

To prove this, observe that, since I is an initial object, the set $\mathrm{Hom}(I,I)$ contains exactly one element, namely id_I, which is the neutral element of this group. Thus, for every $f \in \mathrm{Hom}(X,I)$, we have:

$$f = \mathrm{id}_I \circ f = (\mathrm{id}_I + \mathrm{id}_I) \circ f = \mathrm{id}_I \circ f + \mathrm{id}_I \circ f = f + f.$$

This implies that f is, in fact, the neutral element of the group $\mathrm{Hom}(X,I)$.

Now, suppose T is a terminal object in the category $\mathcal{C}$. We want to show that T is also initial, that is, we want to show that, for every object X in $\mathcal{C}$, the set $\mathrm{Hom}(T,X)$ contains exactly one morphism.

Let X be an object of $\mathcal{C}$. Since $\mathcal{C}$ is assumed to be a preadditive category, we know that $\mathrm{Hom}(T,X)$ is non-empty and, moreover, forms a group. Therefore, showing that $\mathrm{Hom}(T,X)$ contains exactly one element is equivalent to showing that the only morphism in $\mathrm{Hom}(T,X)$ is the neutral element of this group.

To prove this, observe that, since T is terminal, the set $\mathrm{Hom}(T,T)$ contains exactly one element, namely id_T, which is the neutral element of this group. Thus, for every $f \in \mathrm{Hom}(T,X)$, we have:

$$f = f \circ \mathrm{id}_T = f \circ (\mathrm{id}_T + \mathrm{id}_T) = f \circ \mathrm{id}_T + f \circ \mathrm{id}_T = f + f.$$

This implies that f is, in fact, the neutral element of the group $\mathrm{Hom}(T,X)$.

(b) For simplicity of notation, we will prove the case $n = 2$. The general case follows analogously.

Let (P, p_1, p_2) be the product of A_1 and A_2. By definition, this means that $P \in \mathrm{Obj}$, $p_1 \in \mathrm{Hom}(P, A_1)$, $p_2 \in \mathrm{Hom}(P, A_2)$, and that, for every triple (X, f_1, f_2) with $X \in \mathrm{Obj}$, $f_1 \in \mathrm{Hom}(X, A_1)$, and $f_2 \in \mathrm{Hom}(X, A_2)$, there exists a unique $F \in \mathrm{Hom}(X,P)$ such that $p_1 \circ F = f_1$ and $p_2 \circ F = f_2$. We will use this property to construct morphisms $\iota_1 \in \mathrm{Hom}(A_1, P)$ and $\iota_2 \in \mathrm{Hom}(A_2, P)$ such that (P, ι_1, ι_2) is the coproduct of A_1 and A_2.

To construct ι_1 and ι_2, we also use the hypothesis that $\mathcal{C}$ is a preadditive category. In fact, it is this hypothesis that allows us to choose neutral elements $0_{1,2} \in \mathrm{Hom}(A_1, A_2)$ and $0_{2,1} \in \mathrm{Hom}(A_2, A_1)$. Then, for the triple $(A_1, \mathrm{id}_{A_1}, 0_{1,2})$, there exists a unique morphism $\iota_1 \in \mathrm{Hom}(A_1, P)$ such that

$$p_1 \circ \iota_1 = \mathrm{id}_{A_1} \quad \text{and} \quad p_2 \circ \iota_1 = 0_{1,2}.$$

Similarly, for the triple $(A_2, 0_{2,1}, \mathrm{id}_{A_2})$, there exists a unique morphism $\iota_2 \in \mathrm{Hom}(A_2, P)$ such that

$$p_1 \circ \iota_2 = 0_{2,1} \quad \text{and} \quad p_2 \circ \iota_2 = \mathrm{id}_{A_2}\,.$$

To complete the proof of this part, we will show that (P, ι_1, ι_2) is the coproduct of A_1 and A_2. To do this, let X be an object in Obj, let f_1 be a morphism in $\mathrm{Hom}(A_1, X)$, and let f_2 be a morphism in $\mathrm{Hom}(A_2, X)$. The morphism $F \in \mathrm{Hom}(P, X)$ such that $F \circ \iota_1 = f_1$ and $F \circ \iota_2 = f_2$ is explicitly given by $F = (f_1 \circ p_1) + (f_2 \circ p_2)$. In fact,

$$\begin{aligned}
F \circ \iota_1 &= ((f_1 \circ p_1) + (f_2 \circ p_2)) \circ \iota_1 \\
&= (f_1 \circ p_1) \circ \iota_1 + (f_2 \circ p_2) \circ \iota_1 \\
&= f_1 \circ (p_1 \circ \iota_1) + f_2 \circ (p_2 \circ \iota_1) \\
&= f_1 \circ \mathrm{id}_{A_1} + f_2 \circ 0_{1,2} \\
&= f_1
\end{aligned}$$

and

$$\begin{aligned}
F \circ \iota_2 &= ((f_1 \circ p_1) + (f_2 \circ p_2)) \circ \iota_2 \\
&= (f_1 \circ p_1) \circ \iota_2 + (f_2 \circ p_2) \circ \iota_2 \\
&= f_1 \circ (p_1 \circ \iota_2) + f_2 \circ (p_2 \circ \iota_2) \\
&= f_1 \circ 0_{2,1} + f_2 \circ \mathrm{id}_{A_2} \\
&= f_2.
\end{aligned}$$

This shows that (P, ι_1, ι_2) is the coproduct of A_1 and A_2, and completes the proof of this part.

(c) Again, for simplicity of notation, we will prove the case $n = 2$, since the general case is completely analogously.

Let (C, i_1, i_2) be the coproduct of A_1 and A_2. By definition, this means that $C \in \mathrm{Obj}$, $i_1 \in \mathrm{Hom}(A_1, C)$, $i_2 \in \mathrm{Hom}(A_2, C)$, and that for every triple (X, f_1, f_2) with $X \in \mathrm{Obj}$, $f_1 \in \mathrm{Hom}(A_1, X)$, and $f_2 \in \mathrm{Hom}(A_2, X)$, there exists a unique morphism $F \in \mathrm{Hom}(C, X)$ such that $F \circ i_1 = f_1$ and $F \circ i_2 = f_2$. We will use this property to construct morphisms $p_1 \in \mathrm{Hom}(C, A_1)$ and $p_2 \in \mathrm{Hom}(C, A_2)$ such that (C, p_1, p_2) is the product of A_1 and A_2.

To construct p_1 and p_2, we also use the hypothesis that $\mathcal{C}$ is a preadditive category. In fact, it is this hypothesis that allows us to choose neutral elements $0_{1,2} \in \mathrm{Hom}(A_1, A_2)$ and $0_{2,1} \in \mathrm{Hom}(A_2, A_1)$. Then, for the triple $(A_1, 0_{2,1}, \mathrm{id}_{A_1})$, there exists a unique morphism $p_1 \in \mathrm{Hom}(C, A_1)$ such that

$$p_1 \circ i_1 = \mathrm{id}_{A_1} \quad \text{and} \quad p_1 \circ i_2 = 0_{2,1}.$$

Similarly, for the triple $(A_2, \mathrm{id}_{A_2}, 0_{1,2})$, there exists a unique morphism $p_2 \in \mathrm{Hom}(C, A_2)$ such that

$$p_2 \circ i_1 = 0_{1,2} \quad \text{and} \quad p_2 \circ i_2 = \mathrm{id}_{A_2}.$$

To complete the proof, we will show that (C, p_1, p_2) is the product of A_1 and A_2. To do this, let X be an object in Obj, let f_1 be a morphism in $\mathrm{Hom}(X, A_1)$, and let f_2 be a morphism in $\mathrm{Hom}(X, A_2)$. The morphism $F \in \mathrm{Hom}(X, C)$ such that $p_1 \circ F = f_1$ and $p_2 \circ F = f_2$ is explicitly given by $F = (i_1 \circ f_1) + (i_2 \circ f_2)$. In fact,

$$\begin{aligned} p_1 \circ F &= p_1 \circ ((i_1 \circ f_1) + (i_2 \circ f_2)) \\ &= p_1 \circ (i_1 \circ f_1) + p_1 \circ (i_2 \circ f_2) \\ &= (p_1 \circ i_1) \circ f_1 + (p_1 \circ i_2) \circ f_2 \\ &= \mathrm{id}_{A_1} \circ f_1 + 0_{2,1} \circ f_2 \\ &= f_1 \end{aligned}$$

and

$$\begin{aligned} p_2 \circ F &= p_2 \circ ((i_1 \circ f_1) + (i_2 \circ f_2)) \\ &= p_2 \circ (i_1 \circ f_1) + p_2 \circ (i_2 \circ f_2) \\ &= (p_2 \circ i_1) \circ f_1 + (p_2 \circ i_2) \circ f_2 \\ &= 0_{1,2} \circ f_1 + \mathrm{id}_{A_2} \circ f_2 \\ &= f_2. \end{aligned}$$

This shows that (C, p_1, p_2) is the product of A_1 and A_2. □

1.4. Additive categories

Additive categories are a natural generalization of preadditive categories, providing a framework for studying categories with additional structure, such as finite products and coproducts. In this section, we will define additive categories and explore their properties through concrete examples.

Definition 1.4.1. A category $\mathcal{C}$ is said to be *additive* when:

(i) $\mathcal{C}$ is a preadditive category,

(ii) there exists an initial or a terminal object in Obj,

(iii) for every pair of objects $A, B \in \text{Obj}$, their product or their coproduct exists within $\mathcal{C}$.

Recall from Proposition 1.3.5(a) that, in a preadditive category, an object is initial if and only if it is terminal. This explains the *or* in condition (ii) above. Then, recall from Proposition 1.3.5(b) and (c) that, in a preadditive category, the product of two objects exists if and only if their coproduct exists. This explains the *or* in condition (iii) above.

Further, recall from Proposition 1.2.14 that (in any category) the empty product is a terminal object and from Proposition 1.2.20 that the empty coproduct is an initial object. Thus, one could condense conditions (ii) and (iii) above in one condition that requires every finitary product to exist within $\mathcal{C}$.

To better understand the abstract definition of additive categories, we will consider some concrete examples. We will start with the simplest possible category and then explore more complex cases.

Example 1.4.2. The category with one object and one morphism (constructed in Example 1.1.2) is additive. In fact, recall from Example 1.3.2 that this category is preadditive. Further, recall from Example 1.2.5 that the unique object in this category is initial and terminal. Finally, recall from Example 1.2.10 that product of this object with itself is itself and from Example 1.2.16 that the coproduct of this object with itself is itself. This justifies that the category with one object and one morphism is additive.

While the simplest category is additive, not all categories share this property. In the next example, we will see that the category of sets is not additive because it is not even preadditive.

Example 1.4.3. Recall from Example 1.3.3 that the category of sets is not preadditive. Thus, it cannot be an additive category.

Not all preadditive categories are additive. In the next example, we will construct a category that is preadditive but fails to be additive due to the lack of an initial or terminal object.

Example 1.4.4. Now we will construct a category that is preadditive and not additive. First, let Obj consist of a unique element, $\text{Obj} = \{\star\}$, Mor (which is the same as $\operatorname{Hom}(\star,\star)$) consist of two morphisms, $\mathrm{Mor} = \{\mathrm{id}_\star, f\}$, and the composition be given by

$$\mathrm{id}_\star \circ \mathrm{id}_\star = \mathrm{id}_\star, \quad \mathrm{id}_\star \circ f = f \circ \mathrm{id}_\star = f \quad \text{and} \quad f \circ f = f.$$

Then, endow $\operatorname{Hom}(\star,\star)$ with the structure of the abelian group $\mathbb{Z}_2$:

$$\mathrm{id}_\star + \mathrm{id}_\star = f, \quad \mathrm{id}_\star + f = f + \mathrm{id}_\star = \mathrm{id}_\star \quad \text{and} \quad f + f = f.$$

Now, we check that this category is preadditive:

$$\begin{gathered}
(\mathrm{id}_\star + \mathrm{id}_\star) \circ \mathrm{id}_\star = f \circ \mathrm{id}_\star = f = \mathrm{id}_\star + \mathrm{id}_\star = (\mathrm{id}_\star \circ \mathrm{id}_\star) + (\mathrm{id}_\star \circ \mathrm{id}_\star),\\
(\mathrm{id}_\star + \mathrm{id}_\star) \circ f = f \circ f = f = f + f = (\mathrm{id}_\star \circ f) + (\mathrm{id}_\star \circ f),\\
(\mathrm{id}_\star + f) \circ \mathrm{id}_\star = \mathrm{id}_\star \circ \mathrm{id}_\star = \mathrm{id}_\star = \mathrm{id}_\star + f = (\mathrm{id}_\star \circ \mathrm{id}_\star) + (f \circ \mathrm{id}_\star),\\
(\mathrm{id}_\star + f) \circ f = \mathrm{id}_\star \circ f = f = f + f = (\mathrm{id}_\star \circ f) + (f \circ f),\\
(f + f) \circ \mathrm{id}_\star = f \circ \mathrm{id}_\star = f = f + f = (f \circ \mathrm{id}_\star) + (f \circ \mathrm{id}_\star),\\
(f + f) \circ f = f \circ f = f = f + f = (f \circ f) + (f \circ f).
\end{gathered}$$

Notice that what these calculations show is that the triple $(\mathrm{Mor}, +, \circ)$ is in fact a ring. Namely, a ring isomorphic to $\mathbb{Z}_2$ (with the isomorphism being given by $\overline{0} \mapsto f$ and $\overline{1} \mapsto \mathrm{id}_\star$).

To complete this example, notice the only object in this category, $\star$, is not initial (nor terminal), because $\operatorname{Hom}(\star,\star)$ contains *two* morphisms.

1.5. Constructions in categories II

1.5.1. **Equalizers.** In category theory, the concept of equalizer is used to formalize the idea of commonality between two morphisms. More specifically, it provides a way to characterize the universal object through which both morphisms agree.

Definition 1.5.1. Given a category $\mathcal{C}$, two objects $A, B \in \mathrm{Obj}$, and two morphisms $f, g \in \operatorname{Hom}(A, B)$, the *equalizer* of f and g is a pair (E, e) where:

(i) $E \in \mathrm{Obj}$,

(ii) $e \in \operatorname{Hom}(E, A)$,

(iii) $f \circ e = g \circ e$,

(iv) for every object $X \in \mathrm{Obj}$ and morphism $h \in \operatorname{Hom}(X, A)$ such that $f \circ h = g \circ h$, there exists a unique morphism $u : X \to E$ such that $e \circ u = h$.

This first example of equalizer illustrates how the definition of equalizer applies to the situation where we have two equal morphisms, and shows how the equalizer is realized concretely as the domain of this morphism.

Example 1.5.2. Let $\mathcal{C}$ be a category, let A and B be two objects in Obj, and let $f \in \mathrm{Hom}(A, B)$ be a morphism. The equalizer of f and itself is the pair (A, id_A). To justify this claim, notice that:

(i) $A \in \mathrm{Obj}$,

(ii) $\mathrm{id}_A \in \mathrm{Hom}(A, A)$,

(iii) $f \circ \mathrm{id}_A = f \circ \mathrm{id}_A$,

(iv) for every object $X \in \mathrm{Obj}$ and every morphism $h \in \mathrm{Hom}(X, A)$ (such that $f \circ h = f \circ h$), the morphism $u = h$ is the only morphism in $\mathrm{Hom}(X, A)$ such that $\mathrm{id}_A \circ u = h$.

In the next example, we will construct the equalizer of two functions in the category of sets (see Example 1.1.3). In this case, the equalizer is realized as the largest subset in which these two functions agree.

Example 1.5.3. Let A, B be two sets, and let $f, g : A \to B$ be two functions. The equalizer of f and g is the pair (E, e), where E is the set defined by

$$E := \{a \in A \mid f(a) = g(a)\},$$

and e is the inclusion of E into A, that is, the function

$$e : E \to A \qquad \text{defined by} \qquad e(x) = x \quad \text{for all } x \in E.$$

To justify this claim, notice that:

(i) E is a subset of A, that is, an object of the category of sets,

(ii) e is a function, that is, a morphism in the category of sets,

(iii) for every $x \in E$, we have $f(e(x)) = f(x) = g(x) = g(e(x))$,

(iv) for every set X and every function $h : X \to A$ such that $f \circ h = g \circ h$, notice that $h(x) \in E$ for all $x \in X$ (since $f(h(x)) = g(h(x))$). Hence, if one chooses $u : X \to E$ to be defined by $u(x) = h(x)$, then one obtains that $e(u(x)) = e(h(x)) = h(x)$ for all $x \in X$.

With these concrete examples in hand, we now turn to a key result: the uniqueness (up to isomorphism) of equalizers when they exist. That is, the next result establishes that any two equalizers of a given pair of morphisms are isomorphic.

Proposition 1.5.4. Let $\mathcal{C}$ be a category, A, B be two objects in Obj, and f, g be two morphisms in $\mathrm{Hom}(A, B)$. If (E, e) and (E', e') are equalizers of f and g in $\mathcal{C}$, then E is isomorphic to E'.

Proof. From the definition of equalizers and the hypothesis that (E, e) and (E', e') are equalizers of f and g, we know that E and E' are objects of $\mathcal{C}$, that e is a morphism in $\mathrm{Hom}(E, A)$ such that $e \circ f = e \circ g$, and that e' is a morphism in $\mathrm{Hom}(E', A)$ such that $e' \circ f = e' \circ g$. Moreover, since (E, e) and (E', e') are equalizers of f and g, there exist unique morphisms $u \in \mathrm{Hom}(E, E')$ and $u' \in \mathrm{Hom}(E', E)$ such that $e \circ u = e'$ and $e' \circ u' = e$. We will show that u and u' are isomorphisms.

To do this, we begin by substituting the equations into each other:

$$e' = e \circ u = (e' \circ u') \circ u = e' \circ (u' \circ u),$$
$$e = e' \circ u' = (e \circ u) \circ u' = e \circ (u \circ u').$$

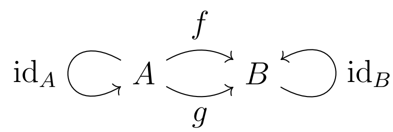
To complete this proof, we will show that $u' \circ u = \mathrm{id}_E$ and $u \circ u' = \mathrm{id}_{E'}$. In fact, since (E, e) is an equalizer of f and g, there exists a unique morphism $v \in \mathrm{Hom}(E, E)$ such that $e \circ v = e$. Since $v = u \circ u'$ and $v = \mathrm{id}_E$ satisfy this condition, it follows that $u \circ u' = \mathrm{id}_E$. Similarly, since (E', e') is an equalizer of f and g, there exists a unique morphism $v' \in \mathrm{Hom}(E', E')$ such that $e' \circ v' = e'$. Since $v' = u' \circ u$ and $v' = \mathrm{id}_{E'}$ satisfy this condition, it follows that $u' \circ u = \mathrm{id}_{E'}$. □

The previous proposition guarantees the uniqueness of equalizers up to isomorphism, but it is important to note that equalizers do not always exist. We finish this subsection with an example that illustrates a situation in which the equalizer does not exist.

Example 1.5.5. Consider a category with two objects, $\mathrm{Obj} = \{A, B\}$, and four morphisms, $\mathrm{Mor} = \{\mathrm{id}_A, f, g, \mathrm{id}_B\}$, where

$$\{\mathrm{id}_A\} = \mathrm{Hom}(A, A), \quad \{f, g\} = \mathrm{Hom}(A, B) \quad \text{and} \quad \{\mathrm{id}_B\} = \mathrm{Hom}(B, B).$$

A diagrammatic picture of this category is the following:



Now, we will show that no equalizer of f and g exists. In fact, if this equalizer (E, e) existed, then E would have to be an object in $\mathrm{Obj} = \{A, B\}$ and e would have to be a morphism in $\mathrm{Hom}(E, A)$ such that $f \circ e = g \circ e$. Since $\mathrm{Hom}(B, A) = \emptyset$, then E would have to be A and e would have to be id_A. However, $f \circ \mathrm{id}_A \neq g \circ \mathrm{id}_A$. This explains why no equalizer of f and g exists. (See Example 1.5.7 for a more subtle situation in which morphisms have no equalizers.)

1.5.2. **Kernels.** In a preadditive category, kernels are a special case of equalizers. They formalize the idea of the "preimage of zero" for a morphism. More specifically, kernels provide a way to characterize the universal object that maps to zero under a given morphism.

Definition 1.5.6. Given a preadditive category $\mathcal{C}$, two objects $A, B \in \mathrm{Obj}$, and a morphism $f \in \mathrm{Hom}(A, B)$, the *kernel* of f is defined to be the equalizer of f and the zero morphism in the abelian group $\mathrm{Hom}(A, B)$.

In a more explicit way, the kernel of a morphism $f \in \mathrm{Hom}(A, B)$ is an object $\ker(f) \in \mathrm{Obj}$ together with a morphism $k \in \mathrm{Hom}(\ker(f), A)$ satisfying the following conditions:

- $f \circ k = 0 \circ k = 0$ in the abelian group $\mathrm{Hom}(\ker(f), B)$,
- for every pair (X, h), where X is an object of $\mathcal{C}$ and $h \in \mathrm{Hom}(X, A)$ is a morphism satisfying $f \circ h = 0 \in \mathrm{Hom}(X, B)$, there exists a unique morphism $u \in \mathrm{Hom}(X, \ker(f))$ such that $k \circ u = h$.

It is important to note that the condition that $\mathcal{C}$ is preadditive is essential in Definition 1.5.6. Without the ability to define zero morphisms, the concept of a kernel cannot be formulated. For example, kernels cannot be defined in the category of sets, as it is not preadditive (see Example 1.3.3).

Additionally, since equalizers are unique up to isomorphism (see Proposition 1.5.4), kernels are also unique up to isomorphism.

In the following examples, we present cases where kernels do and do not exist.

Example 1.5.7. Consider a category $\mathcal{C}$ with one object, $\mathrm{Obj} = \{\star\}$. Recall from Example 1.4.4 that, in this case, for $\mathcal{C}$ to be a preadditive category, one must endow Mor (which is equal to $\mathrm{Hom}(\star,\star)$) with a ring structure, $(\mathrm{Mor}, +, \circ)$. In this example, we will choose the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$. That is, Mor will have four elements,

$$\mathrm{Mor} = \mathrm{Hom}(\star,\star) = \left\{(\overline{0},\overline{0}),\, (\overline{0},\overline{1}),\, (\overline{1},\overline{0}),\, (\overline{1},\overline{1})\right\},$$

its addition will be given by component-wise addition modulo 2, and its composition will be given by component-wise multiplication modulo 2.

Now we will show that the kernel of the morphism $(\overline{0},\overline{1})$ does not exist. In fact, recall that the kernel of $(\overline{0},\overline{1})$ is a pair (K,k), where $K \in \mathrm{Obj}$ and $k \in \mathrm{Hom}(K,\star)$ such that $(\overline{0},\overline{1}) \circ k = (\overline{0},\overline{0})$. Since $\mathrm{Obj} = \{\star\}$, then $K = \star$; and, since

$$(\overline{0},\overline{1}) \circ (\overline{0},\overline{0}) = (\overline{0},\overline{1}) \circ (\overline{1},\overline{0}) = (\overline{0},\overline{0}) \;\text{ and }\; (\overline{0},\overline{1}) \circ (\overline{0},\overline{1}) = (\overline{0},\overline{1}) \circ (\overline{1},\overline{1}) = (\overline{0},\overline{1}),$$

then $k = (\overline{0},\overline{0})$ or $k = (\overline{1},\overline{0})$.

To justify that the pair $(\star, (\overline{0},\overline{0}))$ is not the kernel of $(\overline{0},\overline{1})$, notice that, if we take the pair $(X,h) = (\star, (\overline{1},\overline{0}))$, then $(\overline{0},\overline{1}) \circ h = (\overline{0},\overline{0})$ but there exists no morphism $u \in \mathrm{Hom}(\star,\star)$ such that $(\overline{0},\overline{0}) \circ u = h$. Now, to justify that the pair $(\star, (\overline{1},\overline{0}))$ is not the kernel of $(\overline{0},\overline{1})$, notice that, if we take the pair $(X,h) = (\star, (\overline{0},\overline{0}))$, then $(\overline{0},\overline{1}) \circ h = (\overline{0},\overline{0})$ and there exist *two* morphisms $u \in \mathrm{Hom}(\star,\star)$ such that $(\overline{1},\overline{0}) \circ u = (\overline{0},\overline{0})$, namely, $u = (\overline{0},\overline{0})$ and $u = (\overline{0},\overline{1})$.

This shows that the kernel of the morphism $(\overline{0},\overline{1})$ does not exist. Similarly, one can show that there is no kernel for the morphism $(\overline{1},\overline{0})$ in this category.

In the previous example, we saw a situation where the kernel of a morphism does not exist. In the next example, we turn to a more familiar setting: the category of vector spaces, where kernels always exist and correspond to the classical notion of the kernel of a linear map.

Example 1.5.8. Let $\Bbbk$ be a field (for example, $\mathbb{R}$). Consider the category whose objects are all $\Bbbk$-vector spaces, whose morphisms are all linear transformations between $\Bbbk$-vector spaces, and whose composition is given by the usual composition of linear transformations (or, equivalently, functions). One can check that this structure forms a category, which is locally small.

One can also introduce a structure of abelian group on its hom-sets. Namely, if V and W are $\Bbbk$-vector spaces, then the set $\mathrm{Hom}(V,W)$ is an abelian group when endowed with the function $+ : \mathrm{Hom}(V, W) \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ defined by

$$(T + S)(v) := T(v) + S(v) \ \in \ W \qquad \text{for all} \quad T, S \in \mathrm{Hom}(V, W).$$

Thus, the morphism $0 \in \mathrm{Hom}(V, W)$ is explicitly given by $0(v) = o_W$ for all $v \in V$.

Hence, the kernel of a morphism $T \in \mathrm{Hom}(V, W)$ is the pair $(\ker(T), k)$, where:

- $\ker(T)$ is the usual kernel of linear transformations,

$$\ker(T) = \{v \in V \mid T(v) = o_W\},$$

- k is the inclusion of $\ker(T)$ into V,

$$k : \ker(T) \to V \quad \text{explicitly given by} \quad k(v) = v.$$

In fact, since $\ker(T)$ is a vector subspace of V, it is also a $\Bbbk$-vector space. Moreover, k is a $\Bbbk$-linear transformation, since

$$k(v_1 + \lambda v_2) = v_1 + \lambda v_2 = k(v_1) + \lambda k(v_2).$$

Finally, if (X, h) is a pair where X is a $\Bbbk$-vector space and $h : X \to V$ is a $\Bbbk$-linear transformation such that $T \circ h = 0$, then $h(x) \in \ker(T)$ for all $x \in X$. This implies that the inclusion $u : X \to \ker(T)$ is the unique linear transformation that satisfies $k \circ u = h$.

1.5.3. **Coequalizers.** In category theory, the concept of coequalizer formalizes the idea of identifying elements mapped to the same place by two morphisms. It achieves this by defining a universal object which enforces this identification.

Definition 1.5.9. Given a category $\mathcal{C}$, two objects $A, B \in \mathrm{Obj}$, and two morphisms $f, g \in \mathrm{Hom}(A, B)$, the *coequalizer* of f and g is a pair (Q, q) where:

(i) $Q \in \mathrm{Obj}$,

(ii) $q \in \mathrm{Hom}(B, Q)$,

(iii) $q \circ f = q \circ g$,

(iv) for every object $X \in \mathrm{Obj}$ and morphism $k \in \mathrm{Hom}(B, X)$ such that $k \circ f = k \circ g$, there exists a unique morphism $v \in \mathrm{Hom}(Q, X)$ such that $v \circ q = k$.

The first example of a coequalizer will illustrate how the definition of a coequalizer applies in the case where the two morphisms are equal. In this case the coequalizer is realized concretely as the codomain of this morphism.

Example 1.5.10. Let $\mathcal{C}$ be a category, let A and B be two objects in Obj, and let $f \in \mathrm{Hom}(A, B)$ be a morphism. The coequalizer of f and itself is the pair (B, id_B). To justify this claim, notice that:

(i) $B \in \mathrm{Obj}$,

(ii) $\mathrm{id}_B \in \mathrm{Hom}(B, B)$,

(iii) $\mathrm{id}_B \circ f = \mathrm{id}_B \circ f$,

(iv) for every object $X \in \mathrm{Obj}$ and every morphism $k \in \mathrm{Hom}(B, X)$ (such that $k \circ f = k \circ f$), the morphism $v = k$ is the only morphism in $\mathrm{Hom}(B, X)$ such that $v \circ \mathrm{id}_B = k$.

In the next example, we will construct the coequalizer of two functions in the category of sets (see Example 1.1.3). Here, the coequalizer corresponds to the quotient set that identifies elements that map to the same element under these two functions.

Example 1.5.11. Let A, B be two sets and let $f, g : A \to B$ be two functions. To construct the coequalizer of f and g, consider the equivalence relation in B generated by $f(a) \sim g(a)$ for all $a \in A$. That is:

- for every $b \in B$, we have $b \sim b$,
- if $b = f(a) \sim g(a) = b'$ for some $a \in A$, then $b' = g(a) \sim f(a) = b$,
- if $b = f(a) \sim g(a) = b'$ for some $a \in A$ and $b' = f(a') \sim g(a') = b''$ for some $a' \in A$, then $b \sim b''$.

Then, define Q to be the set of equivalence classes with respect to the equivalence relation $\sim$. If we denote by $[b]$ the equivalence class to which an element $b \in B$ belongs, then:

$$Q = \{[b] \mid b \in B\}.$$

Notice that Q is a set and that there exists a function

$$q : B \to Q \qquad \text{defined by} \qquad q(b) = [b] \quad \text{for all } b \in B.$$

The pair (Q, q) is the coequalizer of f and g.

To justify this claim, notice that:

(i) Q is a quotient set of B, that is, an object of the category of sets,

(ii) q is a function, that is, a morphism in the category of sets,

(iii) since $f(a) \sim g(a)$ for every $a \in A$, we have

$$q(f(a)) = [f(a)] = [g(a)] = q(g(a)),$$

(iv) for every set X and every function $k : B \to X$ such that $k \circ f = k \circ g$, we can define a function $v : Q \to X$ by $v([b]) = k(b)$. In fact, to justify that this is a function, notice that, if $[b] = [b']$, then $k(b) = k(b')$. Moreover, by definition, this function v satisfies $v \circ q = k$. On the other hand, if $v \circ q = k$, then $v([b]) = k(b)$ for all $b \in B$. This shows the uniqueness of v.

Moving on from the concrete examples, we now prove a key result: the uniqueness (up to isomorphism) of coequalizers when they exist. In other words, this next result establishes that any two coequalizers of a given pair of morphisms are isomorphic.

Proposition 1.5.12. Let $\mathcal{C}$ be a category, A, B be two objects in Obj, and f, g be two morphisms in $\mathrm{Hom}(A, B)$. If (Q, q) and (Q', q') are coequalizers of f and g, then Q is isomorphic to Q'.

Proof. From the definition of coequalizers and the hypothesis that (Q, q) and (Q', q') are coequalizers of f and g, we know that Q and Q' are objects of $\mathcal{C}$, that q is a morphism in $\mathrm{Hom}(B, Q)$ such that $q \circ f = q \circ g$, and that q' is a morphism in $\mathrm{Hom}(B, Q')$ such that $q' \circ f = q' \circ g$. Moreover, since (Q, q) and (Q', q') are coequalizers of f and g, there exist unique morphisms $v \in \mathrm{Hom}(Q, Q')$ and $v' \in \mathrm{Hom}(Q', Q)$ such that $v \circ q = q'$ and $v' \circ q' = q$. We will show that v and v' are isomorphisms.

To do this, we begin by substituting the equations above into each other:

$$q' = v \circ q = v \circ (v' \circ q') = (v \circ v') \circ q',$$
$$q = v' \circ q' = v' \circ (v \circ q) = (v' \circ v) \circ q.$$

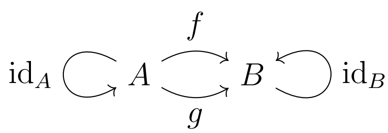
To complete this proof, we will show that $v' \circ v = \mathrm{id}_Q$ and $v \circ v' = \mathrm{id}_{Q'}$. In fact, since (Q, q) is a coequalizer of f and g, there exists a unique morphism $w \in \mathrm{Hom}(Q, Q)$ such that $w \circ q = q$. Since $w = v' \circ v$ and $w = \mathrm{id}_Q$ satisfy this condition, it follows that $v' \circ v = \mathrm{id}_Q$. Similarly, since (Q', q') is a coequalizer of f and g, there exists a unique $w' \in \mathrm{Hom}(Q', Q')$ such that $w' \circ q' = q'$. Since $w' = v \circ v'$ and $w' = \mathrm{id}_{Q'}$ satisfy this condition, it follows that $v \circ v' = \mathrm{id}_{Q'}$. □

The previous proposition guarantees the uniqueness of coequalizers up to isomorphism, but it is important to note that coequalizers do not always exist. We finish this subsection with an example that illustrates a situation in which the coequalizer does not exist.

Example 1.5.13. Consider a category with two objects, $\mathrm{Obj} = \{A, B\}$, and four morphisms, $\mathrm{Mor} = \{\mathrm{id}_A, f, g, \mathrm{id}_B\}$, where

$$\{\mathrm{id}_A\} = \mathrm{Hom}(A, A), \quad \{f, g\} = \mathrm{Hom}(A, B) \quad \text{and} \quad \{\mathrm{id}_B\} = \mathrm{Hom}(B, B).$$

A diagrammatic picture of this category is the following:



Now, we will show that no coequalizer of f and g exists. In fact, if this coequalizer (Q, q) existed, then Q would have to be an object in $\mathrm{Obj} = \{A, B\}$ and q would have to be a morphism in $\mathrm{Hom}(B, Q)$ such that $q \circ f = q \circ g$. Since $\mathrm{Hom}(B, A) = \emptyset$, then Q would have to be B and q would have to be id_B. However, $\mathrm{id}_B \circ f \neq \mathrm{id}_B \circ g$. This explains why no coequalizer of f and g exists. (See Example 1.5.15 for a more subtle situation in which morphisms have no coequalizers.)

1.5.4. **Cokernels.** In a preadditive category, cokernels are a special case of coequalizers. They formalize the idea of "quotient by the image" of a morphism. More specifically, cokernels provide a way to characterize the universal object that maps the image of a given morphism to zero.

Definition 1.5.14. Given a preadditive category $\mathcal{C}$, two objects $A, B \in \mathrm{Obj}$, and a morphism $f \in \mathrm{Hom}(A, B)$, the *cokernel* of f is defined to be the coequalizer of f and the zero morphism in the abelian group $\mathrm{Hom}(A, B)$.

In a more explicit way, the cokernel of a morphism $f \in \mathrm{Hom}(A, B)$ is an object $\mathrm{coker}(f) \in \mathrm{Obj}$ together with a morphism $q \in \mathrm{Hom}(B, \mathrm{coker}(f))$ satisfying the following conditions:

- $q \circ f = q \circ 0 = 0$ in the abelian group $\mathrm{Hom}(A, \mathrm{coker}(f))$,
- for every pair (Y, h), where Y is an object of $\mathcal{C}$ and $h \in \mathrm{Hom}(B, Y)$ is a morphism satisfying $h \circ f = 0 \in \mathrm{Hom}(A, Y)$, there exists a unique morphism $v \in \mathrm{Hom}(\mathrm{coker}(f), Y)$ such that $v \circ q = h$.

It is important to note that the condition that $\mathcal{C}$ is preadditive is also essential in Definition 1.5.14. Without the ability to define zero morphisms, the concept of a cokernel cannot be formulated. For example, cokernels cannot be defined in the category of sets, as it is not preadditive (see Example 1.3.3).

Additionally, since coequalizers are unique up to isomorphism (see Proposition 1.5.12), cokernels are also unique up to isomorphism.

In the following examples, we present cases where cokernels do and do not exist.

Example 1.5.15. Consider a category with one object, $\mathrm{Obj} = \{\star\}$, and with Mor (which is equal to $\mathrm{Hom}(\star, \star)$ in this case) endowed with a ring structure, $(\mathrm{Mor}, +, \circ)$, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see Example 1.5.7 for the details). We will show that the cokernel of the morphism $(\overline{0}, \overline{1})$ does not exist in this category.

In fact, recall that the cokernel of $(\overline{0}, \overline{1})$ is a pair (Q, q), where $Q \in \mathrm{Obj}$ and $q \in \mathrm{Hom}(\star, Q)$ such that $q \circ (\overline{0}, \overline{1}) = (\overline{0}, \overline{0})$. Since $\mathrm{Obj} = \{\star\}$, then $Q = \star$; and, since

$$(\overline{0}, \overline{0}) \circ (\overline{0}, \overline{1}) = (\overline{1}, \overline{0}) \circ (\overline{0}, \overline{1}) = (\overline{0}, \overline{0}) \quad \text{and} \quad (\overline{0}, \overline{1}) \circ (\overline{0}, \overline{1}) = (\overline{1}, \overline{1}) \circ (\overline{0}, \overline{1}) = (\overline{0}, \overline{1}),$$

then $q = (\overline{0}, \overline{0})$ or $q = (\overline{1}, \overline{0})$.

To justify that the pair $(\star, (\overline{0}, \overline{0}))$ is not the cokernel of $(\overline{0}, \overline{1})$, notice that, if we take the pair $(Y, h) = (\star, (\overline{1}, \overline{0}))$, then $h \circ (\overline{0}, \overline{1}) = (\overline{0}, \overline{0})$ but there exists no morphism $v \in \mathrm{Hom}(\star, \star)$ such that $v \circ (\overline{0}, \overline{0}) = (\overline{1}, \overline{0})$. Now, to justify that the pair $(\star, (\overline{1}, \overline{0}))$ is not the cokernel of $(\overline{0}, \overline{1})$, notice that, if we take the pair $(Y, h) = (\star, (\overline{0}, \overline{0}))$, then $h \circ (\overline{0}, \overline{1}) = (\overline{0}, \overline{0})$ and there exist *two* morphisms $v \in \mathrm{Hom}(\star, \star)$ such that $v \circ (\overline{1}, \overline{0}) = (\overline{0}, \overline{0})$, namely, $v = (\overline{0}, \overline{0})$ and $v = (\overline{0}, \overline{1})$.

This shows that the cokernel of the morphism $(\overline{0}, \overline{1})$ does not exist in this category. Similarly, one can show that the morphism $(\overline{1}, \overline{0})$ also has no cokernel in this category.

In the previous example, we saw a situation where the cokernel of a morphism does not exist. In the next example, we turn to a more familiar setting: the category of vector spaces, where cokernels always exist and correspond to the classical notion of the cokernel of a linear map.

Example 1.5.16. Let $\Bbbk$ be a field (for example, $\mathbb{R}$) and consider the category of $\Bbbk$-vector spaces (constructed in Example 1.5.8). The cokernel of a morphism $T \in \mathrm{Hom}(V, W)$ is the pair $(\mathrm{coker}(T), q)$, where:

- $\operatorname{coker}(T)$ is the usual cokernel of the linear transformation T, that is, the quotient of the codomain of T by its image,
$$\operatorname{coker}(T) = W/\operatorname{im}(T);$$
- q is the canonical projection
$$q : W \to \operatorname{coker}(T), \quad \text{explicitly given by} \quad q(w) = [w].$$

In fact, notice that, since $\operatorname{im}(T)$ is a vector subspace of W, then $\operatorname{coker}(T)$ is a quotient vector space. Moreover, q is a $\Bbbk$-linear transformation, since
$$q(w_1 + \lambda w_2) = [w_1 + \lambda w_2] = [w_1] + \lambda[w_2] = q(w_1) + \lambda q(w_2).$$
Finally, if (X, k) is a pair where X is a $\Bbbk$-vector space and $k : W \to X$ is a $\Bbbk$-linear transformation such that $k \circ T = 0$, then k factors through $\operatorname{coker}(T)$, as $\operatorname{im}(T)$ is in its kernel. Thus, the isomorphism theorems from Linear Algebra imply that there exists a unique linear transformation $v : \operatorname{coker}(T) \to X$ satisfying $v \circ q = k$.

1.6. Pre-abelian categories

In this section, we will introduce pre-abelian categories, which are an intermediate step between additive and abelian categories.

Definition 1.6.1. A category $\mathcal{C}$ is said to be *pre-abelian* if $\mathcal{C}$ is an additive category in which the kernel and cokernel of all morphisms exist.

To define pre-abelian categories in more detail, recall that an additive category is defined as a pre-additive category in which finite products or coproducts of its objects (including the empty ones) exist. Then, recall that a pre-additive category is defined as a category in which $\operatorname{Hom}(A, B)$ admits the structure of an abelian group for every pair of objects A, B. Hence, a pre-abelian category is a locally small category for which:

- $\operatorname{Hom}(A, B)$ admits the structure of an abelian group for every pair of objects A, B in $\mathcal{C}$;
- finite products and coproducts of objects in $\mathcal{C}$ exist;
- the kernel and cokernel of every morphism in $\mathcal{C}$ exist.

In the next example, we will show that the category of vector spaces over a field (defined in Example 1.5.8) is pre-abelian. This example illustrates how pre-abelian categories naturally arise in familiar algebraic settings.

Example 1.6.2. Let $\Bbbk$ be a field, and let $\mathcal{C}$ be the category of $\Bbbk$-vector spaces. Recall from Example 1.5.8 that $\mathcal{C}$ is an additive category. Also recall from Example 1.5.8 that the kernel of a morphism in $\mathcal{C}$ is the usual kernel of this morphism, viewed as a linear transformation (as defined in an undergraduate Linear Algebra class). Then, recall from Example 1.5.16 that the cokernel of a morphism (that is, a linear transformation) $T \in \mathrm{Hom}(V, W)$ is the pair (Q, q), where $Q = W/\operatorname{im}(T)$ and q is the canonical projection $q : W \to Q$, explicitly given by $q(w) = w + \operatorname{im}(T)$. This shows that the category of $\Bbbk$-vector spaces is a pre-abelian category.

In the previous example, we showed a very concrete case of a pre-abelian category. However, not all additive categories are pre-abelian. In the next example, we will exhibit an additive category that fails to be pre-abelian because it lacks kernels and cokernels for certain morphisms.

Example 1.6.3. Consider the category with one object, $\mathrm{Obj} = \{\star\}$, and whose morphisms, $\mathrm{Mor} = \mathrm{Hom}(\star, \star)$, are identified with the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$. This category was constructed in detail in Example 1.5.7, where it was also shown to be additive.

Since $\mathrm{Hom}(\star, \star)$ is identified with $\mathbb{Z}_2 \times \mathbb{Z}_2$, we can denote the morphisms in $\mathrm{Hom}(\star, \star)$ by $(\overline{0}, \overline{0})$, $(\overline{0}, \overline{1})$, $(\overline{1}, \overline{0})$, and $(\overline{1}, \overline{1})$. In Example 1.5.7, it was shown that the kernel of the morphism $(\overline{0}, \overline{1})$ does not exist, and in Example 1.5.15, it was shown that the cokernel of the morphism $(\overline{1}, \overline{0})$ also does not exist. Thus, this category is not pre-abelian.

1.7. Constructions in categories III

Monomorphisms and epimorphisms are categorical generalizations of injective and surjective maps, respectively. To extend these notions from sets and vector spaces to general categories, one relies on the "left cancellability", and respectively "right cancellability", properties of these functions. In this section, we define these concepts and provide concrete examples.

1.7.1. **Monomorphisms.**

In this section, we begin with the abstract definition of monomorphisms and then provide concrete examples to illustrate it.

Definition 1.7.1. Given a category $\mathcal{C}$ and two of its objects A and B, a morphism $f \in \mathrm{Hom}(A, B)$ is said to be *monic* (or a *monomorphism*) when, for every object X of $\mathcal{C}$ and every pair of morphisms $g_1, g_2 \in \operatorname{Hom}(X, A)$, we have: $f \circ g_1 = f \circ g_2$ if and only if $g_1 = g_2$.

To illustrate the abstract definition above, we begin by showing that monic morphisms in the category of sets are nothing more than injective maps.

Example 1.7.2. If $\mathcal{C}$ is the category of sets (see Example 1.1.3), then a morphism (that is, a function) $f : A \to B$ is monic if and only if f is injective.

To show this, we begin by assuming that f is an injective map. Now, let X be any set, and let $g_1, g_2 : X \to A$ be functions such that $f \circ g_1 = f \circ g_2$. This means that, $f(g_1(x)) = f(g_2(x))$ for all $x \in X$. Since f is injective, this implies that $g_1(x) = g_2(x)$ for all $x \in X$. This shows that $g_1 = g_2$.

Now, to show the converse, assume that f is monic. To show that f is an injective function, let $a_1, a_2 \in A$ be two elements such that $f(a_1) = f(a_2)$. We will show that $a_1 = a_2$. In fact, consider a set X with only one element, $X = \{x\}$, and define the maps $g_1, g_2 : X \to A$ by $g_1(x) = a_1$ and $g_2(x) = a_2$. Then, by construction, $f \circ g_1 = f \circ g_2$. Now, since f is assumed to be monic, we have that $g_1 = g_2$. This implies $a_1 = g_1(x) = g_2(x) = a_2$.

Now, we will show general examples of monic morphisms that we already considered in previous sections. We begin with isomorphisms.

Example 1.7.3. Let $\mathcal{C}$ be a category, A and B be two objects of $\mathcal{C}$. We want to show that, if $f \in \operatorname{Hom}(A, B)$ is an isomorphism, then it is monic. In fact, recall from Definition 1.2.1 that, if f is an isomorphism, then there exists a morphism $g \in \operatorname{Hom}(B, A)$ such that $g \circ f = \mathrm{id}_A$. Hence, if X is an object of $\mathcal{C}$ and $g_1, g_2 \in \operatorname{Hom}(X, A)$ are morphisms such that $f \circ g_1 = f \circ g_2$, then:

$$g_1 = \mathrm{id}_A \circ g_1 = (g \circ f) \circ g_1 = g \circ (f \circ g_1) = g \circ (f \circ g_2) = (g \circ f) \circ g_2 = \mathrm{id}_A \circ g_1 = g_2.$$

This shows that f is monic.

In the next example, we consider equalizers of morphisms. In fact, we will show how the morphism in the equalizer pair is a monomorphism.

Example 1.7.4. Let $\mathcal{C}$ be a category, and let A and B be two objects of $\mathcal{C}$. Let $f_1, f_2 : A \to B$ be two morphisms, and let (E, e) be the equalizer of f_1 and f_2. Then, $e : E \to A$ is a monomorphism. To justify this claim, let X be any object, and let $g_1, g_2 : X \to E$ be morphisms such that $e \circ g_1 = e \circ g_2$. Since e is the equalizer of f_1 and f_2, we have:

$$f_1 \circ (e \circ g_2) = (f_1 \circ e) \circ g_2 = (f_2 \circ e) \circ g_2 = f_2 \circ (e \circ g_2).$$

Hence, X is an object of $\mathcal{C}$ and $h = (e \circ g_2)$ is a morphism in $\text{Hom}(X, A)$ such that $f_1 \circ h = f_2 \circ h$. By the universal property of the equalizer (see Definition 1.5.1), there exists a unique morphism $u : X \to E$ such that $e \circ u = h$. Since $e \circ g_1 = h = e \circ g_2$, it follows that $g_1 = u = g_2$. This shows that e is a monomorphism.

To finish this section, we will determine which morphisms are monic and which are not monic in a small pre-additive category.

Example 1.7.5. Consider a category $\mathcal{C}$ with a unique object, $\text{Obj} = \{\star\}$. Recall from Example 1.4.4 that a pre-additive structure on $\mathcal{C}$ is equivalent to a ring structure on $\text{Mor} = \text{Hom}(\star, \star)$. In this case, a morphism in $\text{Hom}(\star, \star)$ is monic if and only if it is not a left zero-divisor in the ring $\text{Hom}(\star, \star)$.

We will actually show that a morphism $f \in \text{Hom}(\star, \star)$ is not monic if and only if it is a left zero-divisor in $\text{Hom}(\star, \star)$. To begin, assume that f is not monic. This means that there exist distinct morphisms $g_1, g_2 \in \text{Hom}(\star, \star)$ such that $f \circ g_1 = f \circ g_2$. Since $\text{Hom}(\star, \star)$ is a ring, this means that $f \circ (g_1 - g_2) = 0$; that is, that f is a left zero-divisor. On the other hand, if f is a left zero-divisor in $\text{Hom}(\star, \star)$, then there exists a morphism $g \in \text{Hom}(\star, \star)$ such that $g \neq 0$ and $f \circ g = 0$. Since $f \circ 0 = 0 = f \circ g$ and $g \neq 0$, this implies that f is not a monomorphism.

1.7.2. **Epimorphisms.** In this section, we will begin with the abstract definition of epimorphisms and then provide concrete examples to illustrate it.

Definition 1.7.6. Given a category $\mathcal{C}$ and two of its objects A and B, a morphism $f \in \text{Hom}(A, B)$ is said to be an *epimorphism* when, for every object Y of $\mathcal{C}$ and every pair of morphisms $g_1, g_2 \in \text{Hom}(B, Y)$, we have: $g_1 \circ f = g_2 \circ f$ if and only if $g_1 = g_2$.

To illustrate the abstract definition above, we begin by showing that epimorphisms in the category of sets are nothing more than surjective maps.

Example 1.7.7. If $\mathcal{C}$ is the category of sets (see Example 1.1.3), then a morphism (that is, a function) $f : A \to B$ is epi if and only if f is surjective.

To show this, we begin by assuming that f is a surjective map. Now, let Y be any set, and let $g_1, g_2 : B \to Y$ be functions such that $g_1 \circ f = g_2 \circ f$. This means that, for all $a \in A$, $g_1(f(a)) = g_2(f(a))$. Since f is surjective, every $b \in B$ can be written as $b = f(a)$ for some $a \in A$. Thus, $g_1(b) = g_2(b)$ for all $b \in B$, which shows that $g_1 = g_2$.

Now, to show the converse, assume that f is an epimorphism. Then, consider the set $Y = \{0, 1\}$ and the functions $g_1, g_2 : B \to Y$ defined by:

$$g_1(b) = 1 \quad \text{for all } b \in B \qquad \text{and} \qquad g_2(b) = \begin{cases} 0, & \text{if } b \notin \text{im}(f), \\ 1, & \text{if } b \in \text{im}(f). \end{cases}$$

Notice that Y is an object of $\mathcal{C}$ and g_1, g_2 are morphisms in $\text{Hom}(B, Y)$ such that $g_1 \circ f = g_2 \circ f$. Since f is assumed to be an epimorphism, this implies that $g_1 = g_2$; that is, every $b \in B$ belongs to $\text{im}(f)$. This show that f is surjective.

Now, we will show general examples of epimorphisms that we already considered in previous sections. We begin with isomorphisms.

Example 1.7.8. Let $\mathcal{C}$ be a category, and let A and B be two objects of $\mathcal{C}$. We want to show that, if $f \in \text{Hom}(A, B)$ is an isomorphism, then it is also an epimorphism. In fact, recall from Definition 1.2.1 that, if f is an isomorphism, then there exists a morphism $g \in \text{Hom}(B, A)$ such that $f \circ g = \text{id}_B$. Hence, if Y is an object of $\mathcal{C}$ and $g_1, g_2 \in \text{Hom}(B, Y)$ are morphisms such that $g_1 \circ f = g_2 \circ f$, then:

$$g_1 = g_1 \circ \text{id}_B = g_1 \circ (f \circ g) = (g_1 \circ f) \circ g = (g_2 \circ f) \circ g = g_2 \circ (f \circ g) = g_2 \circ \text{id}_B = g_2.$$

This shows that f is an epimorphism.

In the next example, we consider coequalizers of morphisms. In fact, we will show how the morphism in the coequalizer pair is epi.

Example 1.7.9. Let $\mathcal{C}$ be a category, let A and B be two objects of $\mathcal{C}$, let $f_1, f_2 : A \to B$ be two morphisms, and let (Q, q) be the coequalizer of f_1 and f_2. Then, $q : B \to Q$ is an epimorphism. To justify this claim, let Y be an object of $\mathcal{C}$ and $g_1, g_2 : Q \to Y$ be morphisms such that $g_1 \circ q = g_2 \circ q$. Since q is the coequalizer of f_1 and f_2, we have:

$$(g_1 \circ q) \circ f_1 = g_1 \circ (q \circ f_1) = g_1 \circ (q \circ f_2) = (g_1 \circ q) \circ f_2.$$

Hence, Y is an object of $\mathcal{C}$ and $k := (g_1 \circ q)$ is a morphism in $\text{Hom}(B, Y)$ such that $k \circ f_1 = k \circ f_2$. From the definition of coequalizer, Definition 1.5.9, there exists a unique morphism $v \in \text{Hom}(Q, Y)$ such that $(g_1 \circ q) = k = v \circ q$. Since $g_1 \circ q = g_1 \circ q$, then $g_1 = v = g_2$. This shows that q is an epimorphism.

To finish this section, we will determine which morphisms are epi and which are not epi in a small pre-additive category.

Example 1.7.10. Consider a category $\mathcal{C}$ with a unique object, $\mathrm{Obj} = \{\star\}$. Recall from Example 1.4.4 that a pre-additive structure on $\mathcal{C}$ is equivalent to a ring structure on $\mathrm{Mor} = \mathrm{Hom}(\star, \star)$. In this case, a morphism in $\mathrm{Hom}(\star, \star)$ is epi if and only if it is not a right zero-divisor in the ring $\mathrm{Hom}(\star, \star)$.

We will actually show that a morphism $f \in \mathrm{Hom}(\star, \star)$ is not epi if and only if it is a right zero-divisor in $\mathrm{Hom}(\star, \star)$. To begin, assume that f is not an epimorphism. This means that there exist $g_1, g_2 \in \mathrm{Hom}(\star, \star)$ distinct morphisms such that $g_1 \circ f = g_2 \circ f$. Since $\mathrm{Hom}(\star, \star)$ is a ring, this means that $(g_1 - g_2) \circ f = 0$; that is, that f is a right zero-divisor. On the other hand, if f is a right zero-divisor in $\mathrm{Hom}(\star, \star)$, then there exists a morphism $g \in \mathrm{Hom}(\star, \star)$ such that $g \neq 0$ and $g \circ f = 0$. Since $0 \circ f = 0 = g \circ f$ and $g \neq 0$, this implies that f is not an epimorphism.

1.8. Abelian categories

Abelian categories are a central concept in homological algebra and category theory. They provide a natural setting for studying exact sequences, homology, and cohomology, as they generalize the properties of categories like vector spaces and abelian groups. We will begin this section woth their abstract definition and then illustrate it with concrete examples.

Definition 1.8.1. A category is said to be *abelian* when it is a pre-abelian category, and moreover, every monomorphism is the kernel of a morphism and every epimorphism is the cokernel of a morphism.

To define abelian categories in more detail, recall from Definition 1.6.1 that a category is pre-abelian when it is additive and the kernel and cokernel of every morphism exist. Then, recall from Definition 1.4.1 that a category is additive when it is a pre-additive category in which finite products or coproducts of its objects (including the empty ones) exist. Finally, recall from Definition 1.3.1 that a pre-additive category is one in which $\mathrm{Hom}(A, B)$ admits the structure of an abelian group for every pair of objects A, B, and composition is bilinear. Hence, an abelian category is a locally small category such that:

- $\mathrm{Hom}(A, B)$ admits a structure of an abelian group for every pair of objects A, B in $\mathcal{C}$;
- finite products and coproducts of objects in $\mathcal{C}$ exist;
- the kernel and cokernel of every morphism in $\mathcal{C}$ exist;

- every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism.

To illustrate this abstract definition, we will consider some concrete examples. We begin with the down-to-earth example of vector spaces.

Example 1.8.2. For every field $\mathbb{k}$, the category of $\mathbb{k}$-vector spaces is abelian. To begin justifying this, we recall from Example 1.6.2 that this category is pre-abelian.

Moreover, to show that every monomorphism in this category is the kernel of another morphism, notice that, if a morphism (that is, a linear transformation) $T : V \to W$ is monic, then it is injective (compare with Example 1.7.2), and hence V is isomorphic to $\mathrm{im}(T)$, which is a vector subspace of W. Thus, the pair (V, T) is the kernel of the canonical projection $q : W \to W/\,\mathrm{im}(T)$, explicitly given by $q(w) = w + \mathrm{im}(T)$. This shows that every monomorphism in this category is the kernel of another morphism.

Finally, to show that every epimorphism is the cokernel of another morphism, notice that, if a morphism $T : V \to W$ is epi, then it is surjective (compare with Example 1.7.7). Hence, the Isomorphism Theorems imply that the pair (W, T) is (isomorphic to) the cokernel of the inclusion morphism $\iota : \ker(T) \to V$, explicitly given by $\iota(v) = v$.

Next, we consider the category of abelian groups, whose objects are abelian groups and whose morphisms are group homomorphisms. This is a fundamental example of an abelian category. This example will also work as a review of the concepts introduced so far.

Example 1.8.3. The category of abelian groups, usually denoted by **Ab**, consists of abelian groups and their morphisms. More explicitly, the objects of the category **Ab** are all abelian groups; the morphisms of **Ab** are all group homomorphisms between abelian groups, that is, $\mathrm{Hom}(A, B)$ is the set of all group homomorphisms $A \to B$; and the composition of morphisms in **Ab** is the usual composition of functions.

Recall (from Example A.12) that, if G, H, K are (abelian) groups and $f : G \to H$, $g : H \to K$ are group homomorphisms, then the composition $(g \circ f) : G \to K$ is also a group homomorphism. Also recall (from Example A.12) that, for every (abelian) group, the identity map is a group homomorphism.

The pre-additive structure on the category **Ab** is given by the following structure on the homomorphism sets: given two abelian groups, A and B, the set $\mathrm{Hom}(A, B)$ is an abelian group under operation $+$ defined by:

$$(f+g)(a) := f(a) +_B g(a) \quad \text{for all } f, g \in \mathrm{Hom}(A, B) \text{ and } a \in A$$

(where $+_B$ denotes the abelian group operation in B). Since B is an abelian group, this operation $+$ on $\mathrm{Hom}(A, B)$ is associative and commutative. Moreover, the identity element in $\mathrm{Hom}(A, B)$ is the trivial homomorphism $A \to B$, that is, the function that maps every element in A to the identity element of B. Hence, the opposite of a morphism $f \in \mathrm{Hom}(A, B)$ with respect to this operation $+$ is the morphism $g : A \to B$ defined by $g(a) := -(f(a))$ for all $a \in A$. Finally, the fact that the composition of morphisms is bilinear follows from the fact that:

$$\begin{aligned}((f+g)\circ h)(a) &= (f+g)(h(a)) \\ &= f(h(a)) + g(h(a)) \\ &= (f\circ h)(a) + (g \circ h)(a)\end{aligned}$$

and

$$\begin{aligned}(f\circ (g+h))(a) &= f(g(a)+h(a)) \\ &= f(g(a)) + f(h(a)) \\ &= (f\circ g)(a) + (f \circ h)(a),\end{aligned}$$

for all morphisms of abelian groups $f, g \in \mathrm{Hom}(A, B)$, $h \in \mathrm{Hom}(B, C)$, and for all $a \in A$.

This shows that **Ab** is a pre-additive category. To verify that **Ab** is also an additive category, recall from Definition 1.4.1 that it is enough to construct an initial object within **Ab** and the product of two objects of **Ab**.

The initial object in **Ab** is the trivial group $\{0\}$ (see Example A.6). Since 0 is the only element in this group, for every set A, a function $f : \{0\} \to A$ is uniquely determined by the image of 0. Moreover, since 0 is the identity of the group $\{0\}$, for f to be a group homomorphism, 0 must me mapped to the identity of A. This explains why there exists a unique group homomorphism from $\{0\}$ to any other abelian group.

Now, the product of two abelian groups is constructed as follows. Given two abelian groups, A and B, consider the Cartesian product

$$A \times B = \{(a, b) \mid a \in A, b \in B\},$$

equipped with component-wise addition, that is,

$$(a_1, b_1) + (a_2, b_2) := (a_1 +_{\scriptscriptstyle A} a_2,\ b_1 +_{\scriptscriptstyle B} b_2),$$

where $+_{\scriptscriptstyle A}$ denotes the group operation in A and $+_{\scriptscriptstyle B}$ denotes the group operation in B. Since $+_{\scriptscriptstyle A}$ and $+_{\scriptscriptstyle B}$ are associative and commutative, then $+$ is also associative and commutative. Moreover, the identity element in $A \times B$ is $(0_A, 0_B)$ and the opposite of an element $(a, b) \in A \times B$ is the element $(-a, -b)$. Thus, $A \times B$ is an abelian group.

Then, consider the projection maps $p_A : A \times B \to A$ and $p_B : A \times B \to B$, given explicitly by

$$p_A(a, b) := a \quad \text{and} \quad p_B(a, b) := b \qquad \text{for all} \ \ (a, b) \in A \times B.$$

The fact that p_A and p_B are group homomorphisms follows from the fact that the operation $+$ on $A \times B$ is defined component-wise:

$$\begin{aligned} p_A((a_1, b_1) + (a_2, b_2)) &= p_A(a_1 +_{\scriptscriptstyle A} a_2,\ b_1 +_{\scriptscriptstyle B} b_2) \\ &= a_1 +_{\scriptscriptstyle A} a_2 \\ &= p_A(a_1, b_1) +_{\scriptscriptstyle A} p_A(a_2, b_2) \end{aligned}$$

and

$$\begin{aligned} p_B((a_1, b_1) + (a_2, b_2)) &= p_B(a_1 +_{\scriptscriptstyle A} a_2,\ b_1 +_{\scriptscriptstyle B} b_2) \\ &= b_1 +_{\scriptscriptstyle B} b_2 \\ &= p_B(a_1, b_1) +_{\scriptscriptstyle B} p_B(a_2, b_2). \end{aligned}$$

Now, to show that the triple $(A \times B,\, p_A,\, p_B)$ is the product of A and B in **Ab**, let X be an abelian group and let $f_A : X \to A$ and $f_B : X \to B$ be group homomorphisms. Notice that the function $F : X \to A \times B$ given by $F(x) = (f_A(x), f_B(x))$ is well-defined and satisfies the conditions $p_A \circ F = f_A$ and $p_B \circ F = f_B$. Moreover, F is also a group homomorphism, since

$$\begin{aligned} F(x_1 + x_2) &= (f_A(x_1 + x_2),\ f_B(x_1 + x_2)) \\ &= (f_A(x_1) +_{\scriptscriptstyle A} f_A(x_2),\ f_B(x_1) +_{\scriptscriptstyle B} f_B(x_2)) \\ &= (f_A(x_1), f_B(x_1)) + (f_A(x_2), f_B(x_2))\,, \qquad \text{for all} \ \ x_1, x_2 \in X. \end{aligned}$$

This explains why $(A \times B,\, p_A,\, p_B)$ is the product of A and B in **Ab**.

This shows that **Ab** is an additive category. Now, to verify that **Ab** is also pre-abelian, notice that the kernel of a morphism (that is, a group homomorphism) $f : A \to B$ is the pair (K, k), where:

$$K = \ker(f) := \{a \in A \mid f(a) = e_B\}$$

and

$$k : K \to A \quad \text{is defined by} \quad k(a) = a \;\; \text{for all} \;\; a \in K.$$

To justify this claim, recall from Proposition A.22 that the kernel of a group homomorphism is a subgroup of its domain, and thus, an abelian group with the inherited operation. Moreover, the function k is a group homomorphism, since

$$k(a_1 + a_2) = a_1 + a_2 = k(a_1) + k(a_2) \quad \text{for all} \;\; a_1, a_2 \in K,$$

that satisfies the condition $f \circ k = 0$, since $k(a) \in \ker(f)$ for all $a \in K$.

To complete the justification of the claim that (K, k) is the kernel of f, let X be an abelian group and $h : X \to A$ be a group homomorphism such that $f \circ h = 0$. Notice that this implies that $\operatorname{im}(h) \subseteq K$. Hence, one can define a function $u : X \to K$ by $u(x) := h(x)$ for all $x \in X$. This function is a group homomorphism, since

$$u(x_1 + x_2) = h(x_1 + x_2) = h(x_1) + h(x_2) = u(x_1) + u(x_2) \quad \text{for all} \;\; x_1, x_2 \in X.$$

and moreover, it satisfies the condition $k \circ u = h$ (by construction). Furthermore, if $u : X \to K$ is a group homomorphism that satisfies $k \circ u = h$, then $u(x) = k(u(x)) = h(x)$ for all $x \in X$. This explains why u is the unique homomorphism of groups $X \to K$ that satisfies $k \circ u = h$ and completes the proof that (K, k) is the kernel of f.

Further, to complete the verification that **Ab** is a pre-abelian category, we will construct the cokernel of a morphism in **Ab**. Namely, the cokernel of a morphism $f : A \to B$ is the pair (Q, q) where:

$$Q = B/\operatorname{im}(f) \quad \text{and} \quad q : B \to B/\operatorname{im}(f) \;\; \text{is given by} \;\; q(b) = b + \operatorname{im}(f).$$

To justify this claim, recall (from Proposition A.22) that, since f is a group homomorphism, then $\operatorname{im}(f)$ is a subgroup of B, and since B is an abelian group, then $\operatorname{im}(f)$ is a normal subgroup. Hence, the quotient $B/\operatorname{im}(f)$ is a group when endowed with the structure inherited from B (see Section A.5).

Moreover, the function q is a group homomorphism, since

$$\begin{aligned} q(b_1+b_2) &= (b_1+b_2)+\operatorname{im}(f) \\ &= (b_1+\operatorname{im}(f))+(b_2+\operatorname{im}(f)) \\ &= q(b_1)+q(b_2), \qquad \text{for all} \quad b_1,b_2\in B, \end{aligned}$$

that satisfies the condition $q\circ f=0$, since

$$q(b)=b+\operatorname{im}(f)=0+\operatorname{im}(f) \qquad \text{for all} \quad b\in\operatorname{im}(f).$$

To complete the justification of the claim that (Q,q) is the cokernel of f, let Y be an abelian group and $k:B\to Y$ be a group homomorphism such that $k\circ f=0$. Notice that this implies that $\operatorname{im}(f)\subseteq\ker(k)$. Hence, one can define a function $v:Q\to Y$ by $v(b+\operatorname{im}(f))=k(b)$. Moreover, v is a group homomorphism, since

$$\begin{aligned} v(b_1+b_2+\operatorname{im}(f)) &= k(b_1+b_2) \\ &= k(b_1)+k(b_2) \\ &= v(b_1+\operatorname{im}(f))+v(b_2+\operatorname{im}(f)), \qquad \text{for all} \quad b_1,b_2\in B, \end{aligned}$$

that satisfies the condition $v\circ q=k$, since $v(q(b))=v(b+\operatorname{im}(f))=k(b)$ for all $b\in B$. This explains why (Q,q) is the cokernel of f.

This shows that **Ab** is a pre-abelian category. To finish this example, that is, to finish showing that **Ab** is an abelian category, we will show that every monomorphism in **Ab** is the kernel of another morphism and that every epimorphism in **Ab** is the cokernel of another morphism.

To show that every monomorphism in **Ab** is the kernel of another morphism, notice that, if a morphism (that is, a group homomorphism) $f:G\to H$ is monic, then it is injective (compare with Example 1.7.2). Hence, in this case, G is isomorphic to $\operatorname{im}(f)$. Thus, the pair (G,f) is the kernel of the canonical projection $q:H\to H/\operatorname{im}(f)$, explicitly given by $q(h)=h+\operatorname{im}(f)$. This shows that every monomorphism in **Ab** is the kernel of another morphism.

Finally, to show that every epimorphism in **Ab** is the cokernel of another morphism, notice that, if a morphism $f:G\to H$ is epi, then it is surjective (compare with Example 1.7.7). Hence, the Isomorphism Theorems imply that the pair (H,f) is (isomorphic to) the cokernel of the inclusion $\iota:\ker(f)\to G$, explicitly given by $\iota(g)=g$.

This shows that **Ab** is an abelian category and finishes this example.

To finish this section and the first part of these notes, we will exhibit an example of a pre-abelian category that fails to be abelian.

Example 1.8.4. Let $\mathcal{C}$ be the category defined as follows:

- The objects of $\mathcal{C}$ are the free abelian groups $\{0\}, \mathbb{Z}, \mathbb{Z}^2, \dots, \mathbb{Z}^n, \dots$
- For two objects $\mathbb{Z}^n$ and $\mathbb{Z}^m$ of $\mathcal{C}$, the morphisms in $\mathrm{Hom}_{\mathcal{C}}(\mathbb{Z}^n, \mathbb{Z}^m)$ are homomorphisms of groups $\mathbb{Z}^n \to \mathbb{Z}^m$.
- The composition of morphisms is given by standard composition.

This is a pre-abelian category which is not an abelian category.

The fact that $\mathcal{C}$ is a small category follows from: the fact that $\mathrm{Obj}(\mathcal{C})$ is a set in bijection with the set of natural numbers, the fact that $\mathrm{Hom}_{\mathcal{C}}(\mathbb{Z}^n, \mathbb{Z}^m)$ is a set in bijection with the set of $m \times n$ integral matrices (for every pair of objects $n, m \in \mathrm{Obj}(\mathcal{C})$), the fact that composition is associative, and the fact that identity functions are the identity morphisms.

The preadditive structure on $\mathcal{C}$ is given by pointwise addition of group homomorphisms. In fact, since $\mathbb{Z}$ is an abelian group when endowed with its usual addition (see Example A.2), for every pair of objects, $\mathbb{Z}^n, \mathbb{Z}^m \in \mathrm{Obj}(\mathcal{C})$, the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(\mathbb{Z}^n, \mathbb{Z}^m)$ is also an abelian group when endowed with addition defined point-by-point. The fact that the composition of morphisms distributes over this addition follows from the usual distributive laws of multiplications over sums of integers (see Example B.3).

To show that $\mathcal{C}$ is an additive category, we will verify that $\{0\}$ is an initial and final object of $\mathcal{C}$ and that $\mathbb{Z}^{n+m}$ is the product of the abelian groups $\mathbb{Z}^n$ and $\mathbb{Z}^m$. The fact that $\{0\}$ is both an initial and final object of $\mathcal{C}$ follows from the fact that the only homomorphisms of groups $\{0\} \to \mathbb{Z}^n$ and $\mathbb{Z}^n \to \{0\}$ are the constant zero homomorphisms (for any $n \geq 0$).

To verify that $\mathbb{Z}^{n+m}$ is the product of the abelian groups $\mathbb{Z}^n$ and $\mathbb{Z}^m$, denote by p_n the projection of an $n+m$-tuple in $\mathbb{Z}^{n+m}$ on its first n-coordinates,

$$p_n : \mathbb{Z}^{n+m} \to \mathbb{Z}^n, \quad \text{given by} \quad p_n(z_1, z_2, \dots, z_{n+m}) = (z_1, z_2, \dots, z_n),$$

and denote by p_m the projection of an $n+m$-tuple in $\mathbb{Z}^{n+m}$ on its latter m-coordinates,

$$p_m : \mathbb{Z}^{n+m} \to \mathbb{Z}^m, \quad \text{given by} \quad p_m(z_1, z_2, \dots, z_{n+m}) = (z_{n+1}, z_{n+2}, \dots, z_{n+m}).$$

Then, suppose that $\mathbb{Z}^r$ is an object of $\mathcal{C}$ and $q_n : \mathbb{Z}^r \to \mathbb{Z}^n$ and $q_m : \mathbb{Z}^r \to \mathbb{Z}^m$ are homomorphisms of groups. By construction, a homomorphism of groups

$q : \mathbb{Z}^r \to \mathbb{Z}^{n+m}$ satisfies $p_n \circ q = q_n$ and $p_m \circ q = q_m$ if and only if it is defined by

$$q(\mathbf{z}) = (q_n(\mathbf{z}), q_m(\mathbf{z})).$$

This implies that $\mathbb{Z}^{n+m}$ is the product of the abelian groups $\mathbb{Z}^n$ and $\mathbb{Z}^m$, and as a consequence, shows that $\mathcal{C}$ is an additive category.

To show that $\mathcal{C}$ is a pre-abelian category, we will show that the usual kernel of a homomorphism of groups $f : \mathbb{Z}^n \to \mathbb{Z}^m$ is its kernel viewed as a morphism of $\mathcal{C}$ and that the quotient of the group $\mathbb{Z}^m / \operatorname{im}(f)$ by its torsion is the cokernel of f viewed as a morphism of $\mathcal{C}$.

To show that the usual kernel of a homomorphism of groups $f : \mathbb{Z}^n \to \mathbb{Z}^m$,

$$\{\mathbf{z} \in \mathbb{Z}^n \mid f(\mathbf{z}) = 0\},$$

is the kernel of f viewed as a morphism of $\mathcal{C}$, recall from Proposition A.22 that this kernel is a subgroup of $\mathbb{Z}^n$. Since a subgroup of a free abelian group is also a free abelian group (see [Hun80, Theorem II.1.6]), then $\ker(f)$ is an object of $\mathcal{C}$. Hence, we can consider the pair (K, k), where K is the usual kernel of f and k is the inclusion of K inside $\mathbb{Z}^n$. By construction, k is a homomorphism of groups such that $f \circ k = 0$. Moreover, notice that, if K' is a free abelian group and $k' : K' \to \mathbb{Z}^n$ is a homomorphism of groups such that $f \circ k' = 0$, then $\operatorname{im}(k') \subseteq K$. This implies that we can define a function $u : K' \to K$ by $u(x) = k'(x)$. By construction, u is a homomorphism of groups satisfying $k \circ u = k'$. The uniqueness of u follows from the fact that k is the inclusion (a monomorphism). This shows that the usual kernel of a homomorphism of groups $\mathbb{Z}^n \to \mathbb{Z}^m$ is its kernel in the category $\mathcal{C}$.

To construct the cokernel of a morphism $f : \mathbb{Z}^n \to \mathbb{Z}^m$ in $\mathcal{C}$, we will use once again the fact that a subgroup of a free abelian group is a free abelian group [Hun80, Theorem II.1.6]. More specifically, we will use the fact that there exist $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_m \in \mathbb{Z}^m$, a natural number $r \leq m$, and integers $d_1, d_2, \dots, d_r \in \mathbb{Z}$, such that $\mathbb{Z}^m$ is generated by $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_m$ and $\operatorname{im}(f)$ is the subgroup of $\mathbb{Z}^m$ generated by $d_1\mathbf{z}_1, d_2\mathbf{z}_2, \dots, d_r\mathbf{z}_r$. We will thus show that the cokernel of f is given by the pair (C, c), where $C = \mathbb{Z}^{m-r}$ and $c : \mathbb{Z}^m \to \mathbb{Z}^{m-r}$ is given by the projection on the latter $m - r$ coordinates with respect to $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_m$,

$$c(n_1\mathbf{z}_1 + n_2\mathbf{z}_2, \dots + n_m\mathbf{z}_m) = (n_{r+1}, n_{r+2}, \dots, n_m).$$

By construction, C is an object of $\mathcal{C}$ and c is a morphism of $\mathcal{C}$ such that $c \circ f = 0$. Moreover, if C' is another object of $\mathcal{C}$ and $c' : \mathbb{Z}^m \to C'$ is another homomorphism of groups such that $c' \circ f = 0$, then $\operatorname{im}(f) \subseteq \ker(c')$. This implies that $c'(\mathbf{z}_i) = 0$, since $d_i c'(\mathbf{z}_i) = c'(d_i \mathbf{z}_i) = 0$, for all $i \in \{1, 2, \ldots, r\}$. Thus, the function $u : \mathbb{Z}^{m-r} \to C'$ defined by

$$u(n_1, n_2, \ldots, n_{m-r}) = c'(n_1 \mathbf{z}_{r+1} + n_2 \mathbf{z}_{r+2} + \cdots + n_{m-r} \mathbf{z}_r),$$

is a homomorphism of groups such that $u \circ c = c'$. The uniqueness of u follows from the fact that c is surjective (an epimorphism). This shows that the pair (C, c) is the cokernel of the morphism $f : \mathbb{Z}^n \to \mathbb{Z}^m$ is $\mathcal{C}$.

To verify that $\mathcal{C}$ is not an abelian category, we will show that the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(z) = 2z$ is a monomorphism which is not the kernel of any morphism in $\mathcal{C}$. To verify that f is a monomorphism, let $g : \mathbb{Z}^n \to \mathbb{Z}$ be a homomorphism of groups such that $f \circ g = 0$. This means that

$$2g(\mathbf{z}) = f(g(\mathbf{z})) = 0 \quad \text{ for all } \mathbf{z} \in \mathbb{Z}^n,$$

and implies that $g(\mathbf{z}) = 0$ for all $\mathbf{z} \in \mathbb{Z}^n$. That is, if $g : \mathbb{Z}^n \to \mathbb{Z}$ is a homomorphism of groups such that $f \circ g = 0$, then $g = 0$. This means that f is a monomorphism.

To conclude this example, we will verify that f is not the kernel of any morphism in $\mathcal{C}$. To do that, we will show that, if $\phi : \mathbb{Z} \to \mathbb{Z}^n$ is a homomorphism of groups such that $f \circ \phi = 0$, then $\phi = 0$. Hence, if f were to be the kernel of any morphism, this morphism would have to be the constant zero morphism; whose kernel is the identity morphism, not f (see Example 1.5.2). In fact, suppose $\phi : \mathbb{Z} \to \mathbb{Z}^n$ is a homomorphisms of groups such that $\phi \circ f = 0$. By construction, this means that $2\phi(z) = \phi(2z) = \phi(f(z)) = 0$ for all $z \in \mathbb{Z}$, which implies that $\phi = 0$. This finishes the proof that $\mathcal{C}$ is a pre-abelian category that is not abelian.

Part II

Functors

In the first part of these notes, we concentrated on the internal structure of categories, analysing objects, morphisms, and composition within a single axiomatic framework. In this second part, we introduce the concept of a *functor* as the fundamental means of relating categories to one another.

Functors can be understood as structure-preserving maps between categories. We begin this part by presenting the abstract definition and some foundational examples of functors. Then, we proceed to explore several types of functors that will be used in the other parts of these notes: faithful, full, fully faithful, exact and adjoint functors. Along the way, we also present new constructions: limits, colimits, images of morphisms and exact sequences.

Since we will deal with relations between categories, we will need to distinguish between distinct categories in this part. Thus, if necessary, given a category $\mathcal{C}$, we will denote its objects by $\mathrm{Obj}(\mathcal{C})$, its morphisms by $\mathrm{Mor}(\mathcal{C})$, and its composition by $\circ_{\mathcal{C}}$. Further, given two objects X and Y of $\mathcal{C}$, we may denote the class of morphisms between them by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

2.1. Functors

A functor is a structure-preserving relation between categories. It provides a way to relate objects and morphisms of one category to objects and morphisms of another category, while respecting composition and identities. We begin this section with the abstract definition of functors and then follow it up with a few examples.

Definition 2.1.1. Given two categories, $\mathcal{C}$ and $\mathcal{D}$, a *functor* is a relation F that assigns an object of $\mathcal{D}$ to each object of $\mathcal{C}$ and a morphism of $\mathcal{D}$ to each morphism of $\mathcal{C}$ in such a way that:

(i) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Obj}(\mathcal{C})$,

(ii) $F(f \circ_{\mathcal{C}} g) = F(f) \circ_{\mathcal{D}} F(g)$ for all $f, g \in \mathrm{Mor}(\mathcal{C})$ such that $f \circ_{\mathcal{C}} g \in \mathrm{Mor}(\mathcal{C})$.

Notice that, in the definition above, we denote by the same symbol (F) the relation between objects and morphisms. If one were to differentiate them by denoting the relation on objects by F^{Obj} and the relation on morphisms by F^{Mor}, then the first condition above would read

$$F^{\text{Mor}}(\text{id}_X) = \text{id}_{F^{\text{Obj}}(X)} \,.$$

Also notice that, implicit in the second condition above, is the condition that, if $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, then $F^{\text{Mor}}(f) \in \text{Hom}_{\mathcal{D}}(F^{\text{Obj}}(X), F^{\text{Obj}}(Y))$.

Thus, functors can be thought of as "homomorphisms of categories", much like group homomorphisms preserve group structure. To better understand their definition, we present some examples below.

Example 2.1.2. For any category $\mathcal{C}$, the *identity functor* $\text{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined to be given by:

- $\text{Id}_{\mathcal{C}}(X) = X$ for every object X of $\mathcal{C}$,
- $\text{Id}_{\mathcal{C}}(f) = f$ for every morphism f of $\mathcal{C}$.

Since this functor identifies every object and morphism with itself, it trivially satisfies the conditions (i) and (ii) in Definition 2.1.1.

The identity functor given in the example above is the simplest example of a functor. In the next example, we will construct a functor that is not the identity one.

Example 2.1.3. Consider a category $\mathcal{C}$ with two objects, $\text{Obj}(\mathcal{C}) = \{A, B\}$, and three morphisms, $\text{Mor}(\mathcal{C}) = \{\text{id}_A, f, \text{id}_B\}$, where $f \in \text{Hom}_{\mathcal{C}}(A, B)$. In this case, a functor $F : \mathcal{C} \to \mathcal{C}$ must assign:

$$F(A) \in \{A, B\} \quad \text{and} \quad F(B) \in \{A, B\},$$
$$F(\text{id}_A) = \text{id}_{F(A)}, \quad F(f) \in \text{Hom}_{\mathcal{C}}(F(A), F(B)) \quad \text{and} \quad F(\text{id}_B) = \text{id}_{F(B)} \,.$$

For instance, one possible functor that is not the identity one is obtained by choosing:

$$F(A) = F(B) = A \quad \text{and} \quad F(\text{id}_A) = F(f) = F(\text{id}_B) = \text{id}_A \,.$$

On the other hand, notice that there is no functor that assigns

$$F(A) = B \quad \text{and} \quad F(B) = A,$$

since there is no morphism in $\text{Hom}_{\mathcal{C}}(B, A)$ to serve as $F(f)$.

A particularly important class of functors are the so-called *Hom-functors*, which capture the morphism sets in a category.

Example 2.1.4. Let $\mathcal{C}$ be a locally small category. For a fixed object $A \in \mathcal{C}$, the Hom functor

$$\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Sets}$$

is defined by assigning:

- To every object X of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(A, -)(X)$ defined by $\mathrm{Hom}_{\mathcal{C}}(A, X)$,
- For every two objects X and Y of $\mathcal{C}$ and every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the function $\mathrm{Hom}_{\mathcal{C}}(A, -)(f) : \mathrm{Hom}_{\mathcal{C}}(A, X) \to \mathrm{Hom}_{\mathcal{C}}(A, Y)$ that maps a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(A, X)$ to the morphism $(f \circ g) \in \mathrm{Hom}_{\mathcal{C}}(A, Y)$.

The fact that these assignments define a functor follows from the properties of the identity morphisms and the associativity of the composition of morphisms.

Notice that in the example above, we fixed the object A in the first component of Hom. A natural question is whether it is also possible to fix an object in the second component of Hom. As we show in the next example, the answer is '*yes*', but with a little difference.

Example 2.1.5. Let $\mathcal{C}$ be a locally small category. Its opposite category $\mathcal{C}^{\mathrm{op}}$ is the one whose objects are the same as those of $\mathcal{C}$ and whose morphisms are reversed, that is, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ for every two objects X and Y of $\mathcal{C}$.

For a fixed object $A \in \mathcal{C}$, the *contravariant* Hom functor

$$\mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$$

is defined by assigning:

- To every object X of $\mathcal{C}^{\mathrm{op}}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, A)$,
- To every morphism $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y)$ between objects X and Y of $\mathcal{C}^{\mathrm{op}}$, the function $\mathrm{Hom}_{\mathcal{C}}(f, A) : \mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(Y, A)$ that maps a morphism $h \in \mathrm{Hom}_{\mathcal{C}}(X, A)$ to the morphism $(h \circ f) \in \mathrm{Hom}_{\mathcal{C}}(Y, A)$.

The fact that these assignments define a functor also follows from the properties of the identity morphisms and the associativity of the composition of morphisms.

Now that we have seen examples of functors, we close this section with a result that states two fundamental properties of functors: they can be composed and this composition is associative.

Proposition 2.1.6. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be categories, let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ and $H : \mathcal{C} \to \mathcal{D}$ be functors.

(a) The assignment $(G \circ F) : \mathcal{A} \to \mathcal{C}$, defined by
- $(G \circ F)(X) = G(F(X))$ for every object X of $\mathcal{A}$,
- $(G \circ F)(f) = G(F(f))$ for every morphism f of $\mathcal{A}$,

is also a functor.

(b) The functors $H \circ (G \circ F)$ and $(H \circ G) \circ F$ are equal.

Proof. (a) To prove that the assignment $(G \circ F)$ is a functor, we need to verify that it satisfies conditions (i) and (ii) from Definition 2.1.1. In fact:

(i) For every object $X \in \mathrm{Obj}(\mathcal{A})$, we have:

$$\begin{aligned}(G \circ F)(\mathrm{id}_X) &= G(F(\mathrm{id}_X)) \\ &= G(\mathrm{id}_{F(X)}) \\ &= \mathrm{id}_{G(F(X))} \\ &= \mathrm{id}_{(G \circ F)(X)} \,.\end{aligned}$$

(ii) For every three objects X, Y, Z of $\mathcal{A}$, and for every two morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{A}}(Y, Z)$, we have:

$$\begin{aligned}(G \circ F)(g \circ f) &= G(F(g \circ f)) \\ &= G(F(g) \circ F(f)) \\ &= G(F(g)) \circ G(F(f)) \\ &= (G \circ F)(g) \circ (G \circ F)(f).\end{aligned}$$

This shows that $G \circ F$ is indeed a functor from $\mathcal{A}$ to $\mathcal{C}$.

(b) To show that the functors $H \circ (G \circ F)$ and $(H \circ G) \circ F$ are equal, we must verify that their assignments of objects and morphisms are the same. In fact:

- For every object X of $\mathcal{A}$:

$$\begin{aligned}(H \circ (G \circ F))(X) &= H((G \circ F)(X)) \\ &= H(G(F(X))) \\ &= (H \circ G)(F(X)) \\ &= ((H \circ G) \circ F)(X).\end{aligned}$$

- For every morphism f of $\mathcal{A}$:

$$\begin{aligned}(H \circ (G \circ F))(f) &= H((G \circ F)(f)) \\ &= H(G(F(f))) \\ &= (H \circ G)(F(f)) \\ &= ((H \circ G) \circ F)(f).\end{aligned}$$

This shows that $H \circ (G \circ F) = (H \circ G) \circ F$. □

2.2. Faithful, Full and Fully Faithful Functors

In the remaining sections of these notes, we will define several particularly important types of functors. In this section, we will introduce faithful, full and fully faithful functors. We will also provide several examples to illustrate these concepts.

Definition 2.2.1. Given two locally small categories, $\mathcal{C}$ and $\mathcal{D}$, a functor

$$F : \mathcal{C} \to \mathcal{D}$$

is said to be:

- *faithful* when $F : \mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \to \mathrm{Hom}_{\mathcal{D}}(F(c_1), F(c_2))$ is injective for all pair of objects c_1, c_2 of $\mathcal{C}$;
- *full* when $F : \mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \to \mathrm{Hom}_{\mathcal{D}}(F(c_1), F(c_2))$ is surjective for all pair of objects c_1, c_2 of $\mathcal{C}$;
- *fully faithful* when $F : \mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \to \mathrm{Hom}_{\mathcal{D}}(F(c_1), F(c_2))$ is bijective for all pair of objects c_1, c_2 of $\mathcal{C}$.

We will illustrate the definitions above with a few examples. We will begin by showing that the identity functor is fully faithful.

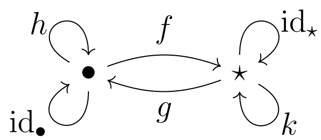


FIGURE 2.2.1. Category constructed in Example 2.2.3

Example 2.2.2. Let $\mathcal{C}$ be a locally small category. Recall that the identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is given by

$$F(c) = c \quad \text{for all } c \in \mathrm{Obj}(\mathcal{C}) \qquad \text{and} \qquad F(f) = f \quad \text{for all } f \in \mathrm{Mor}(\mathcal{C}).$$

This functor is fully faithful. In fact, for every pair of objects c, d of $\mathcal{C}$, we have that $F : \mathrm{Hom}_{\mathcal{C}}(c, d) \to \mathrm{Hom}_{\mathcal{C}}(F(c), F(d))$ is the identity function. Since identity functions are bijections, we conclude that F is a fully faithful functor.

Now, we will consider a locally small category and construct a two functors between them, including an example of a fully faithful functor that is not the identity one.

Example 2.2.3. Let $\mathcal{C}$ be the category with two objects, $\mathrm{Obj}(\mathcal{C}) = \{\bullet, \star\}$, six morphisms, $\mathrm{Mor}(\mathcal{C}) = \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) \cup \mathrm{Hom}_{\mathcal{C}}(\bullet, \star) \cup \mathrm{Hom}_{\mathcal{C}}(\star, \bullet) \cup \mathrm{Hom}_{\mathcal{C}}(\star, \star)$, where

$$\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) = \{\mathrm{id}_\bullet, h\}, \quad \mathrm{Hom}_{\mathcal{C}}(\bullet, \star) = \{f\},$$
$$\mathrm{Hom}_{\mathcal{C}}(\star, \bullet) = \{g\}, \quad \mathrm{Hom}_{\mathcal{C}}(\star, \star) = \{\mathrm{id}_\star, k\},$$

and the compositions of these morphisms are given by

$$\begin{gathered}
\mathrm{id}_\bullet \circ \mathrm{id}_\bullet = \mathrm{id}_\bullet, \quad \mathrm{id}_\bullet \circ h = h, \quad h \circ g = g, \\
h \circ \mathrm{id}_\bullet = h, \quad h \circ h = \mathrm{id}_\bullet, \quad \mathrm{id}_\bullet \circ g = g, \\
f \circ \mathrm{id}_\bullet = f, \quad f \circ h = f, \quad f \circ g = k, \\
g \circ f = h, \quad g \circ \mathrm{id}_\star = g, \quad g \circ k = g, \\
\mathrm{id}_\star \circ f = f, \quad \mathrm{id}_\star \circ \mathrm{id}_\star = \mathrm{id}_\star, \quad \mathrm{id}_\star \circ k = k, \\
k \circ f = f, \quad k \circ \mathrm{id}_\star = k, \quad k \circ k = \mathrm{id}_\star .
\end{gathered}$$

A diagrammatic representation of this category is shown in Figure 2.2.1.

Now we will proceed to construct two functors from this category to itself. We begin by constructing a functor that is neither faithful nor full. Consider any functor $F : \mathcal{C} \to \mathcal{C}$ such that

$$F(\bullet) = \bullet \quad \text{and} \quad F(\star) = \bullet.$$

In order to be a functor, notice that $F(\mathrm{Mor}(\mathcal{C})) \subseteq \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) = \{\mathrm{id}_\bullet, h\}$. In particular, if we choose $c_1 = \bullet$ and $c_2 = \star$, we see that the function

$$F : \mathrm{Hom}_{\mathcal{C}}(\bullet, \star) \to \mathrm{Hom}_{\mathcal{C}}(F(\bullet), F(\star))$$

cannot be surjective. This means that no such functor can be full. In order to guarantee that F is also not a faithful functor, let

$$F(f) = F(g) = F(h) = F(k) = F(\mathrm{id}_\bullet) = F(\mathrm{id}_\star) = \mathrm{id}_\bullet\,.$$

Since $\mathrm{id}_\bullet$ is an identity morphism, this assignment defines a functor. And, since there exists no morphism $\phi \in \mathrm{Mor}(\mathcal{C})$ such that $F(\phi) = h$, we see that, when we choose $c_1 = c_2 = \bullet$, the function $F : \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) \to \mathrm{Hom}_{\mathcal{C}}(F(\bullet), F(\bullet))$ is not injective. This shows that F is also not faithful.

Next, we will construct a fully faithful functor $G : \mathcal{C} \to \mathcal{C}$ that is different from the identity. Define G by assigning:

- $G(\bullet) = \star$ and $G(\star) = \bullet$,
- $G(\mathrm{id}_\bullet) = \mathrm{id}_\star$, $G(h) = k$, $G(f) = g$, $G(g) = f$, $G(\mathrm{id}_\star) = \mathrm{id}_\bullet$, and $G(k) = h$.

To verify that G is a functor, begin by noticing that

$$G(\mathrm{id}_\bullet) = \mathrm{id}_\star = \mathrm{id}_{G(\bullet)} \quad \text{and} \quad G(\mathrm{id}_\star) = \mathrm{id}_\bullet = \mathrm{id}_{G(\star)}\,.$$

Then, notice that:

$$
\begin{aligned}
G(\mathrm{id}_\bullet \circ \mathrm{id}_\bullet) = G(\mathrm{id}_\bullet) = \mathrm{id}_\star = \mathrm{id}_\star \circ \mathrm{id}_\star = G(\mathrm{id}_\bullet) \circ G(\mathrm{id}_\bullet),\\
G(\mathrm{id}_\bullet \circ h) = G(h) = k = \mathrm{id}_\star \circ k = G(\mathrm{id}_\bullet) \circ G(h),\\
G(h \circ g) = G(g) = f = k \circ f = G(h) \circ G(g),\\
G(h \circ \mathrm{id}_\bullet) = G(h) = k = k \circ \mathrm{id}_\star = G(h) \circ G(\mathrm{id}_\bullet),\\
G(h \circ h) = G(\mathrm{id}_\bullet) = \mathrm{id}_\star = k \circ k = G(h) \circ G(h),\\
G(\mathrm{id}_\bullet \circ g) = G(g) = f = \mathrm{id}_\star \circ f = G(\mathrm{id}_\bullet) \circ G(g),\\
G(f \circ \mathrm{id}_\bullet) = G(f) = g = g \circ \mathrm{id}_\star = G(f) \circ G(\mathrm{id}_\bullet),\\
G(f \circ h) = G(f) = g = g \circ k = G(f) \circ G(h),\\
G(f \circ g) = G(k) = h = g \circ f = G(f) \circ G(g),\\
G(g \circ f) = G(h) = k = f \circ g = G(g) \circ G(f),\\
G(g \circ \mathrm{id}_\star) = G(g) = f = f \circ \mathrm{id}_\bullet = G(g) \circ G(\mathrm{id}_\star),\\
G(g \circ k) = G(g) = f = f \circ h = G(g) \circ G(k),\\
G(\mathrm{id}_\star \circ f) = G(f) = g = \mathrm{id}_\bullet \circ g = G(\mathrm{id}_\star) \circ G(f),\\
G(\mathrm{id}_\star \circ \mathrm{id}_\star) = G(\mathrm{id}_\star) = \mathrm{id}_\bullet = \mathrm{id}_\bullet \circ \mathrm{id}_\bullet = G(\mathrm{id}_\star) \circ G(\mathrm{id}_\star),\\
G(\mathrm{id}_\star \circ k) = G(k) = h = \mathrm{id}_\bullet \circ h = G(\mathrm{id}_\star) \circ G(k),\\
G(k \circ f) = G(f) = g = h \circ g = G(k) \circ G(f),\\
G(k \circ \mathrm{id}_\star) = G(k) = h = h \circ \mathrm{id}_\bullet = G(k) \circ G(\mathrm{id}_\star),\\
G(k \circ k) = G(\mathrm{id}_\star) = \mathrm{id}_\bullet = h \circ h = G(k) \circ G(k).
\end{aligned}
$$

This implies that G is indeed a functor. Finally, to verify that G is fully faithful, notice that

$$
G : \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) \to \mathrm{Hom}_{\mathcal{C}}(\star, \star), \quad G : \mathrm{Hom}_{\mathcal{C}}(\bullet, \star) \to \mathrm{Hom}_{\mathcal{C}}(\star, \bullet),
$$
$$
G : \mathrm{Hom}_{\mathcal{C}}(\star, \bullet) \to \mathrm{Hom}_{\mathcal{C}}(\bullet, \star) \quad \text{and} \quad G : \mathrm{Hom}_{\mathcal{C}}(\star, \star) \to \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet)
$$

are all bijections. This shows that G is a fully faithful functor, different from the identity.

In the examples above, we only considered functors from a category to itself. We close this section with an example where we consider a functor between different categories.

Example 2.2.4. Let $\Bbbk$ be a field and denote by $\mathcal{C}$ the category of vector spaces over $\Bbbk$; that is, the category whose objects are vector spaces over $\Bbbk$, whose morphisms are linear transformations between these vector spaces, and whose composition is given by the usual composition of linear transformations (see Example 1.5.8). Then, let $\mathcal{D}$ be the category of sets (see Example 1.1.3). Next, define a functor $F : \mathcal{C} \to \mathcal{D}$ in the following way:

- to each vector space $(V, +, \cdot)$, assign its underlying set, $F(V, +, \cdot) = V$;
- to each linear transformation $T : V \to W$ between vector spaces, assign its underlying function (which is also denoted by T).

This is a faithful functor which is not full.

In fact, to verify that F is a functor, notice that $F(\mathrm{id}_{(V,+,\cdot)}) = \mathrm{id}_V$ for every $\Bbbk$-vector space $(V, +, \cdot)$. Also notice that $F(T \circ S) = F(T) \circ F(S)$ because the composition of linear transformations is defined to be the composition of their underlying functions. This justifies the claim that F is a functor.

Now, to verify that the functor F is faithful, notice that, for every pair of linear transformations $T, S : V \to W$, we have: $F(T) = F(S)$ if and only if $T = S$, because the equality of linear transformations is by definition by the equality of their underlying functions. Finally, to verify that F is not full, recall that not every function between sets is a linear transformation. For instance, any function $f : V \to W$ such that $f(o) \neq o$ is not a linear transformation $V \to W$. Since $\Bbbk$ is assumed to be a field, one can construct one such function whenever W is different from the 0-dimensional $\Bbbk$-vector space $\{o\}$. This justifies the claim that the functor F is not full.

2.3. Constructions in Categories IV

2.3.1. **Limits.** Limits provide a unified framework for constructing universal objects in category theory, generalizing other concepts introduced earlier, such as products and equalizers. The concept of a limit captures the idea of an object that approximates a diagram of objects and morphisms in an optimal way by satisfying a universal property. In this subsection, we define limits abstractly and illustrate this concept through a progression of examples.

Definition 2.3.1 (limits)**.** Given two categories, $\mathcal{C}$ and $\mathcal{I}$, a *diagram in* $\mathcal{C}$ *of shape* $\mathcal{I}$ is defined to be a functor $D : \mathcal{I} \to \mathcal{C}$. A *cone* over a diagram D of shape $\mathcal{I}$ is defined to be an object $c \in \mathrm{Obj}(\mathcal{C})$ together with a family of morphisms $\{\psi_i \in \mathrm{Hom}_{\mathcal{C}}(c, D(i)) \mid i \in \mathrm{Obj}(\mathcal{I})\}$ such that $D(f) \circ \psi_i = \psi_j$ for every morphism $f \in \mathrm{Hom}_{\mathcal{I}}(i, j)$. A *limit* of a diagram D is a cone $(c, \{\psi_i \in \mathrm{Hom}_{\mathcal{C}}(c, D(i))\}_i)$

such that, for every cone $(c', \{\psi'_i \in \mathrm{Hom}_{\mathcal{C}}(c', D(i))\}_i)$ over D, there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{C}}(c', c)$ such that $\psi'_i = \psi_i \circ u$ for all $i \in \mathrm{Obj}(\mathcal{I})$. In this case, the limit is denoted by $\varprojlim D$ or $\lim D$.

This abstract definition may seem daunting at first, but it encapsulates a simple idea: the limit is an object equipped with morphisms to each object in the diagram, and these morphisms are compatible with the structure of the diagram in a universal way. Any other object with compatible morphisms must factor uniquely through the limit. To build intuition, we begin by constructing the limit of a diagram in the category of sets.

Example 2.3.2. Let **Sets** be the category of sets (see Example 1.1.3) and $\mathcal{I}$ be a small category. A diagram in **Sets** of shape $\mathcal{I}$ is a functor $D : \mathcal{I} \to \mathbf{Sets}$. That is, for every object $j \in \mathrm{Obj}(\mathcal{I})$, we assign a set D_i, and to every morphism $f \in \mathrm{Hom}_{\mathcal{I}}(i, j)$, we assign a function $D_f : D_i \to D_j$.

Given a diagram D of shape $\mathcal{I}$ in **Sets**, a cone over D consists of a set C and a family of functions $\{\psi_i : C \to D_i \mid i \in \mathrm{Obj}(\mathcal{I})\}$, such that $D_f \circ \psi_i = \psi_j$ for every morphism $f \in \mathrm{Hom}_{\mathcal{I}}(i, j)$. Given one such diagram, to construct its limit, first consider the product $\prod_{i \in \mathrm{Obj}(\mathcal{I})} D_i$. Then, define C to be the subset consisting of the tuples $(d_i)_i \in \prod_{i \in \mathrm{Obj}(\mathcal{I})} D_i$, such that $D_f(d_i) = d_j$ for every morphism $f \in \mathrm{Hom}_{\mathcal{I}}(i, j)$. In other words, C consists of all tuples that respect the transition maps of the diagram. Next, define a function $\pi_i : C \to D_i$ by setting $\pi_i((d_i)_i) = d_i$ (the i-th coordinate of the tuple) for every $i \in \mathrm{Obj}(\mathcal{I})$. Notice that $D_f \circ \pi_i = \pi_j$ automatically from the construction of C.

Now, to show that the cone $(C, \{\pi_i\}_i)$ is the limit of the diagram D, notice that, for every cone $(C', \{\psi'_j : C' \to D_i\}_i)$ over D, we can define a function $u : C' \to C$ by setting $u(c') = (\psi'_i(c'))_i$ for every element $c' \in C'$. The fact that u is well-defined follows from the fact that the functions ψ'_i satisfy the condition $D_f \circ \psi'_i = \psi'_j$ by construction. The fact that $\psi'_i = \pi_i \circ u$ for all $i \in \mathrm{Obj}(\mathcal{I})$ follows from the definitions of u and π_i:

$$\pi_i\left(u(c')\right) = \pi_i\left((\psi'_i(c'))_i\right) = \psi'_i(c') \quad \text{for all } c' \in C'.$$

Finally, the uniqueness of u follows from the equation above: the i-th coordinate of $u(c')$ must be $\pi_i(u(c')) = \psi'_i(c')$, for every $i \in \mathrm{Obj}(\mathcal{I})$.

This shows that the pair $(C, \{\pi_i\}_i)$ is the limit of the diagram D in the category of **Sets**. This provides a very concrete construction for this limit.

This explicit construction in the category of sets illustrates the general principle: limits can be built from products by imposing compatibility conditions. We now examine how limits specialize to other familiar constructions. First, we identify terminal objects with limits. This will provide us with a further array of concrete examples of limits.

Example 2.3.3. Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be the empty category (with no objects and no morphisms). A diagram $D : \mathcal{I} \to \mathcal{C}$ is necessarily the empty functor. A cone over this empty diagram consists simply of an object $c \in \mathcal{C}$ (with no morphisms to specify, since there are no objects in the diagram). Hence, the limit of the empty diagram is a terminal object in $\mathcal{C}$ (see Definition 1.2.4).

Having seen the degenerate case of the empty diagram and terminal objects, we now consider the situation in which the limit of a diagram corresponds to the product.

Example 2.3.4. Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be the category with two objects, $\mathrm{Obj}(\mathcal{I}) = \{1, 2\}$, and two morphisms, $\mathrm{Mor}(\mathcal{I}) = \{\mathrm{id}_1, \mathrm{id}_2\}$. In this case, a diagram $D : \mathcal{I} \to \mathcal{C}$ consists simply of two objects, $D(1) = c_1$ and $D(2) = c_2$, since $D(\mathrm{id}_1) = \mathrm{id}_{c_1}$ and $D(\mathrm{id}_2) = \mathrm{id}_{c_2}$ automatically. Hence, a cone over this diagram consists of an object c together with morphisms $\psi_1 \in \mathrm{Hom}_{\mathcal{C}}(c, c_1)$ and $\psi_2 \in \mathrm{Hom}_{\mathcal{C}}(c, c_2)$. Thus, if the product of c_1 and c_2 exists in $\mathcal{C}$, it is the limit of this diagram. In fact, from the definition of product (Definition 1.2.9), we have that, for every object c and every pair of morphisms $\psi_1 \in \mathrm{Hom}_{\mathcal{C}}(c, c_1)$ and $\psi_2 \in \mathrm{Hom}_{\mathcal{C}}(c, c_2)$, there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{C}}(c, c_1 \times c_2)$ such that $\psi_1 = \pi_1 \circ u$ and $\psi_2 = \pi_2 \circ u$. This is precisely the property satisfied by the limit of the diagram D.

Having identified limits with products, we now show how limits generalize another important construction: equalizers.

Example 2.3.5. Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be the category with two objects, $\mathrm{Obj}(\mathcal{I}) = \{1, 2\}$, and four morphisms, $\mathrm{Mor}(\mathcal{I}) = \{\mathrm{id}_1, \mathrm{id}_2, f, g\}$, where f and g are morphisms in $\mathrm{Hom}_{\mathcal{I}}(1, 2)$. A diagram $D : \mathcal{I} \to \mathcal{C}$ consists of two objects, $c_1 = D(1)$ and $c_2 = D(2)$, and two morphisms, $D(f), D(g) \in \mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$. Hence, a cone over this diagram consists of an object c of $\mathcal{C}$ and two morphisms, $\psi_1 \in \mathrm{Hom}_{\mathcal{C}}(c, c_1)$ and $\psi_2 \in \mathrm{Hom}_{\mathcal{C}}(c, c_2)$, such that $D(f) \circ \psi_1 = \psi_2 = D(g) \circ \psi_1$. In particular, notice that: $D(f) \circ \psi_1 = D(g) \circ \psi_1$ and ψ_2 is uniquely determined by ψ_1.

Thus, if the equalizer of $D(f)$ and $D(g)$ exists in $\mathcal{C}$, the limit of this diagram is this equalizer. In fact, recall from Definition 1.5.1 that the equalizer of $D(f)$ and $D(g)$ consists of an object e and two morphisms, $\psi \in \mathrm{Hom}_{\mathcal{C}}(e, c_1)$ and $D(f) \circ \psi \in \mathrm{Hom}_{\mathcal{C}}(e, c_2)$, such that: $D(f) \circ \psi = D(g) \circ \psi$ and, for every object e' and every morphism $\psi' \in \mathrm{Hom}_{\mathcal{C}}(e', c_1)$ satisfying $D(f) \circ \psi' = D(g) \circ \psi'$, there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{C}}(e', e)$ such that $\psi' = \psi \circ u$. This is precisely the universal property satisfied by the limit of the diagram D in $\mathcal{C}$.

These examples show that limits generalize constructions in categories that we had previously introduced and provide an array of examples of limits, as well as, examples in which limits do not exist. To close this section, we will consider a more structured type of limit that arises frequently and from inverse systems indexed by partially ordered sets.

Example 2.3.6. Let $\mathcal{C}$ be a category and I be a poset, that is, I is a set endowed with a partial order $\leq$. An *inverse system* in $\mathcal{C}$ is a family of objects $\{c_i \mid i \in I\}$ and a family of morphisms $\{f_{ij} \in \mathrm{Hom}_{\mathcal{C}}(c_i, c_j) \mid j \leq i \in I\}$ such that:

- $f_{ii} = \mathrm{id}_{c_i}$ for all $i \in I$,
- $f_{jk} \circ f_{ij} = f_{ik}$ for all $k \leq j \leq i \in I$.

We can realize direct systems in $\mathcal{C}$ as functors from a category $\mathcal{I}$ to $\mathcal{C}$, that is, as diagrams of shape $\mathcal{I}$ in $\mathcal{C}$. In fact, let $\mathcal{I}$ be the small category with object set I and morphisms determined by the partial order $\leq$: $\mathrm{Hom}_{\mathcal{I}}(i, j)$ has one morphism (which we will denote simply by $i \to j$) if and only if $j \leq i$ in I. Hence, a diagram $D : \mathcal{I} \to \mathcal{C}$ is the assignment of: an object $D(i)$ to each object $i \in \mathrm{Obj}(\mathcal{I})$, and a morphism $D(i \to j) \in \mathrm{Hom}_{\mathcal{C}}(D(i), D(j))$ to each morphism $i \to j$. Furthermore, these morphisms must satisfy the following conditions (see Definition 2.1.1):

- $D(\mathrm{id}_i) = \mathrm{id}_{D(i)}$ for all $i \in I$,
- $D(j \to k) \circ D(i \to j) = D(i \to k)$ for all $k \leq j \leq i \in I$.

In this particular case, the limit of the directed system is defined to be the limit of the corresponding diagram. More specifically, this limit is a pair $(c, \{\psi_i \in \mathrm{Hom}_{\mathcal{C}}(c, c_i) \mid i \in I\})$ such that: $f_{ij} \circ \psi_i = \psi_j$ for all $j \leq i$ and, if $(c', \{\psi'_i \in \mathrm{Hom}_{\mathcal{C}}(c', c_i) \mid i \in I\})$ is such that $f_{ij} \circ \psi'_i = \psi'_j$ for all $j \leq i$, then there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{C}}(c', c)$ such that $\psi'_i = \psi_i \circ u$ for all $i \in I$.

2.3.2. **Colimits.** Colimits also provide a unified framework for constructing universal objects in category theory and generalize other concepts such as coproducts and coequalizers. In this subsection, we define colimits abstractly and illustrate this concept through a progression of examples.

Definition 2.3.7 (colimits)**.** Given two categories, $\mathcal{C}$ and $\mathcal{I}$, a *diagram in* $\mathcal{C}$ *of shape* $\mathcal{I}$ is defined to be a functor $D : \mathcal{I} \to \mathcal{C}$. Given one such diagram D, a *cocone* over D is defined to be an object c of $\mathcal{C}$ together with a family of morphisms $\{\phi_i \in \mathrm{Hom}_{\mathcal{C}}(D(i), c) \mid i \in \mathrm{Obj}(\mathcal{I})\}$, such that $\phi_j \circ D(f) = \phi_i$ for every morphism $f \in \mathrm{Hom}_{\mathcal{I}}(i, j)$. The *colimit* of a diagram D is defined to be a cocone $(c, \{\phi_i \in \mathrm{Hom}_{\mathcal{C}}(D(i), c)\}_i)$ such that, for every cocone $(c', \{\phi'_i \in \mathrm{Hom}_{\mathcal{C}}(D(i), c')\}_i)$, there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{C}}(c, c')$ such that $\phi'_i = u \circ \phi_i$ for all $i \in \mathrm{Obj}(\mathcal{I})$. In this case, the colimit is denoted by $\varinjlim D$ or $\operatorname{colim} D$.

This abstract definition may seem daunting at first, but it encapsulates a simple idea: the colimit is an object equipped with morphisms from each object in the diagram, and these morphisms are compatible with the structure of the diagram in a universal way. Any other object with compatible morphisms must factor uniquely from the colimit. To build intuition, we begin by constructing the colimit of a diagram in the category of sets.

Example 2.3.8. Let **Sets** be the category of sets (see Definition 1.1.3) and let $\mathcal{I}$ be a small category. A diagram in **Sets** of shape $\mathcal{I}$ is a functor $D : \mathcal{I} \to \mathbf{Sets}$, that is, a set D_i is assigned to each object i of $\mathcal{I}$, and a function $D_f : D_i \to D_j$ is assigned to each morphism $f \in \mathrm{Hom}_{\mathcal{I}}(i, j)$.

Given a diagram D in **Sets**, a cocone over D consists of a set C and a family of functions $\{\phi_i : D_i \to C \mid i \in \mathrm{Obj}(\mathcal{I})\}$, such that $\phi_j \circ D_f = \phi_i$ for every morphism $f \in \mathrm{Hom}_{\mathcal{I}}(i, j)$. To construct the colimit of one such diagram, first consider the disjoint union $\bigsqcup_{i \in \mathrm{Obj}(\mathcal{I})} D_i$. Then, define $\sim$ as the smallest equivalence relation on this disjoint union such that $d_i \sim d_j$ when there exists a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(i, j)$ such that $D_f(d_i) = d_j$. Next, define C to be the quotient set $\left(\bigsqcup_{i \in \mathrm{Obj}(\mathcal{I})} D_i\right)/\!\sim$ and denote the equivalence class in C of an element c in D_i by $[c]$. Finally, for each $i \in \mathrm{Obj}(\mathcal{I})$, define $\iota_i : D_i \to C$ to be the function given by $\iota_i(d_i) = [d_i]$. Notice that $\iota_j \circ D_f = \iota_i$ automatically from the construction of the equivalence relation.

To show that the cocone $(C, \{\iota_i\}_i)$ is the colimit of the diagram D, begin by noticing that, for every cocone $(C', \{\phi'_i : D_i \to C'\}_i)$, we can define a function

$u : C \to C'$ by setting $u([d_i]) = \phi'_i(d_i)$. To see that u is well-defined, note that if $d_i \sim d_j$, then the cocone conditions $\phi'_j \circ D_f = \phi'_i$ ensure that $\phi'_i(d_i) = \phi'_j(d_j)$, so the value of u does not depend on the choice of representative. Then, notice that $\phi'_i = u \circ \iota_i$ for all $i \in \mathrm{Obj}(\mathcal{I})$ by the definitions of u and ι_i:

$$u(\iota_i(d_i)) = u([d_i]) = \phi'_i(d_i) \quad \text{ for all } d_i \in D_i \text{ and all } i \in \mathrm{Obj}(\mathcal{I}).$$

Finally, notice that u is completely determined by the equation above.

This shows that the pair $(C, \{\iota_i\}_i)$ is the colimit of the diagram D in the category **Sets**. This provides a very concrete construction for colimits.

This explicit construction in the category of sets illustrates the general principle: colimits can be built from coproducts. We now examine how colimits specialize to other familiar constructions. First, we identify initial objects with colimits. This will provide us with a further array of examples of colimits.

Example 2.3.9. Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be the empty category (with no objects and no morphisms). A diagram $D : \mathcal{I} \to \mathcal{C}$ is necessarily the empty functor. A cocone over this empty diagram consists simply of an object $c \in \mathcal{C}$ (with no morphisms to specify, since there are no objects in the diagram). Hence, the colimit of the empty diagram is an initial object in $\mathcal{C}$: an object c such that for every object $c' \in \mathcal{C}$, there exists a unique morphism $c \to c'$.

Having seen the degenerate case of the empty diagram and initial objects, we now consider the situation in which the colimit of a diagram corresponds to the coproduct.

Example 2.3.10. Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be the discrete category with two objects, $\mathrm{Obj}(\mathcal{I}) = \{1, 2\}$, and two morphisms, $\mathrm{Mor}(\mathcal{I}) = \{\mathrm{id}_1, \mathrm{id}_2\}$. A diagram $D : \mathcal{I} \to \mathcal{C}$ consists simply of two objects, $D(1) = c_1$ and $D(2) = c_2$, since $D(\mathrm{id}_1) = \mathrm{id}_{c_1}$ and $D(\mathrm{id}_2) = \mathrm{id}_{c_2}$. Hence, a cocone over this diagram consists of an object c together with morphisms $\phi_1 \in \mathrm{Hom}_{\mathcal{C}}(c_1, c)$ and $\phi_2 \in \mathrm{Hom}_{\mathcal{C}}(c_2, c)$. Thus, if the coproduct of c_1 and c_2 exists in $\mathcal{C}$, then it is the colimit of this diagram. In fact, from the definition of coproducts (Definition 1.2.15), we have that, for every object c and every pair of morphisms $\phi_1 \in \mathrm{Hom}_{\mathcal{C}}(c_1, c)$ and $\phi_2 \in \mathrm{Hom}_{\mathcal{C}}(c_2, c)$, there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{C}}(c_1 \sqcup c_2, c)$ such that $\phi_1 = u \circ \iota_1$ and $\phi_2 = u \circ \iota_2$.

Having identified colimits with coproducts, we now show how colimits generalize another important construction: coequalizers.

Example 2.3.11. Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be the category with two objects, $\mathrm{Obj}(\mathcal{I}) = \{1, 2\}$, and four morphisms, $\mathrm{Mor}(\mathcal{I}) = \{\mathrm{id}_1, \mathrm{id}_2, f, g\}$, where f and g are morphisms in $\mathrm{Hom}_{\mathcal{I}}(1, 2)$. A diagram $D : \mathcal{I} \to \mathcal{C}$ consists of two objects, $c_1 = D(1)$ and $c_2 = D(2)$, and two morphisms, $D(f), D(g) \in \mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$. A cocone over this diagram consists of an object c of $\mathcal{C}$ and two morphisms, $\phi_1 \in \mathrm{Hom}_{\mathcal{C}}(c_1, c)$ and $\phi_2 \in \mathrm{Hom}_{\mathcal{C}}(c_2, c)$, such that $\phi_2 \circ D(f) = \phi_1 = \phi_2 \circ D(g)$. Notice that, we have that: $\phi_2 \circ D(f) = \phi_2 \circ D(g)$ and ϕ_1 is uniquely determined by ϕ_2.

Thus, if the coequalizer of the morphisms $D(f)$ and $D(g)$ exists in $\mathcal{C}$, then it is the colimit of this diagram. In fact, recall from Definition 1.5.9 that the coequalizer of $D(f)$ and $D(g)$ consists of an object c of $\mathcal{C}$ and a morphism $\pi \in \mathrm{Hom}_{\mathcal{C}}(c_2, c)$, such that: $\pi \circ D(f) = \pi \circ D(g)$ and, for every other object c' and every morphism $\pi' \in \mathrm{Hom}_{\mathcal{C}}(c_2, c')$ satisfying $\pi' \circ D(f) = \pi' \circ D(g)$, there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{C}}(c, c')$ such that $\pi' = u \circ \pi$. This is exactly the universal property satisfied by the colimit of the diagram D.

These examples show that colimits recover other constructions in categories and provide an array of examples of colimits. To close this section, we will consider a more structured type of colimit that arises frequently and from direct systems indexed by partially ordered sets.

Example 2.3.12. Let $\mathcal{C}$ be a category and I be a poset, that is, let I is a set endowed with a partial order $\leq$. A *direct system* in $\mathcal{C}$ is a family of objects $\{c_i \mid i \in I\}$ and a family of morphisms $\{f_{ij} \in \mathrm{Hom}_{\mathcal{C}}(c_i, c_j) \mid i \leq j \in I\}$ such that:

- $f_{ii} = \mathrm{id}_{c_i}$ for all $i \in I$,
- $f_{jk} \circ f_{ij} = f_{ik}$ for all $i \leq j \leq k \in I$.

We can realize direct systems in $\mathcal{C}$ as functors from a category $\mathcal{I}$ to $\mathcal{C}$. More precisely, let $\mathcal{I}$ be the small category with object set I and morphisms determined by the partial order $\leq$: $\mathrm{Hom}_{\mathcal{I}}(i, j)$ has one morphism (which we will denote simply by $i \to j$) if and only if $i \leq j$ in I. Hence, a functor $D : \mathcal{I} \to \mathcal{C}$ is the assignment of an object $D(i) \in \mathrm{Obj}(\mathcal{C})$ to each object $i \in I$, and a morphism $D(i \to j) : D(i) \to D(j)$ in $\mathcal{C}$ to each morphism $i \to j$ in $\mathcal{I}$. Furthermore, these morphisms must satisfy the following conditions:

- $D(\mathrm{id}_i) = \mathrm{id}_{D(i)}$ for all $i \in I$,
- $D(i \to k) = D(j \to k) \circ D(i \to j)$ for all $i \leq j \leq k \in I$.

In this particular case, the colimit of the direct system is defined to be the colimit of the corresponding diagram. More specifically, this colimit is a pair $(c, \{\phi_i \in \mathrm{Hom}_{\mathcal{C}}(c_i, c)\}_i)$ such that: $\phi_j \circ f_{ij} = \phi_i$ for all $i \leq j$ in I and, if $(c', \{\phi'_i \in \mathrm{Hom}_{\mathcal{C}}(c_i, c')\}_i)$ is another cocone such that $\phi'_j \circ f_{ij} = \phi'_i$ for all $i \leq j$ in I, then there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{C}}(c, c')$ such that $\phi'_i = u \circ \phi_i$ for all $i \in I$.

2.3.3. **Images of Morphisms.** In this section, we define the image of a morphism in an abstract category, provide some examples, and show how the abstract definition captures the essential range of a morphism. In familiar categories, we will see that this definition agrees with the usual notion of image.

Definition 2.3.13 (image of morphism)**.** Given a category $\mathcal{C}$, the *image of a morphism* $f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ is a triple $(\mathrm{im}(f), m, e)$ satisfying the following property:

- $\mathrm{im}(f)$ is an object of $\mathcal{C}$,
- $m \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{im}(f), b)$ is a monomorphism,
- $e \in \mathrm{Hom}_{\mathcal{C}}(a, \mathrm{im}(f))$ is such that $f = m \circ e$,
- if x is an object of $\mathcal{C}$ and $e' \in \mathrm{Hom}_{\mathcal{C}}(a, x)$ and $m' \in \mathrm{Hom}_{\mathcal{C}}(x, b)$ are morphisms such that m' is a monomorphism and $f = m' \circ e'$, then there exists a unique morphism $u : \mathrm{im}(f) \to x$ such that $m = m' \circ u$.

As is usual with definitions given in terms of universal properties, when an image of a morphism exists, it is unique up to isomorphism. This abstract definition encodes the intuition that the image should be the smallest subobject through which f factors: since m is a monomorphism, $\mathrm{im}(f)$ can be viewed as a subobject of b, and the universal property ensures that any other monomorphic factorization of f factors through $\mathrm{im}(f)$. However, not all categories have images for all morphisms. We will illustrate both situations through examples. To begin, we show that the image of a morphism in the category of sets is the usual image of a function.

Example 2.3.14. Let **Sets** be the category of sets (see Example 1.1.3). For any function between sets, $f : A \to B$, its image (in the categorical sense) coincides with the usual set-theoretic image,

$$\mathrm{im}(f) = \{b \in B \mid \text{there exists } a \in A \text{ such that } b = f(a)\}.$$

To justify this claim, notice that:

- $\operatorname{im}(f)$ is a set.
- The inclusion map, $m : \operatorname{im}(f) \to B$ given by $m(b) = b$ for all $b \in \operatorname{im}(f)$, is a monomorphism in **Sets** (see Example 1.7.2).
- The restriction of f, that is, the map $e : A \to \operatorname{im}(f)$ given by $e(a) = f(a)$ for all $a \in A$, is a function such that $f = m \circ e$.
- Suppose X is a set and $e' : A \to X$ and $m' : X \to B$ are functions such that m' is injective (that is, monic, see Example 1.7.2) and $f = m' \circ e'$. This means that $f(a) = m'(e'(a))$ for all $a \in A$. Hence, we can define a function $u : \operatorname{im}(f) \to X$ as follows: for each $b \in \operatorname{im}(f)$, choose any $a \in A$ such that $f(a) = b$, and set $u(b) = e'(a)$. To see that u is well-defined, suppose $f(a_1) = f(a_2) = b$ for some $a_1, a_2 \in A$. Then
$$m'(e'(a_1)) = f(a_1) = f(a_2) = m'(e'(a_2)).$$
Since m' is injective, we have $e'(a_1) = e'(a_2)$, so the choice of a does not matter, or equivalently, u is well-defined. By definition, we have
$$m'(u(f(a))) = m'(e'(a)) = f(a) = m(f(a)),$$
that is, $m' \circ u = m$. The uniqueness of u follows from the fact that m' is a monomorphism. In fact, for every function $u' : \operatorname{im}(f) \to X$ such that $m' \circ u' = m$, we have $m' \circ u' = m' \circ u$, and as a consequence, $u' = u$.

Next, we show a case where the image can be computed explicitly.

Example 2.3.15. Let $\mathcal{C}$ be a category and a be an object of $\mathcal{C}$. The image of id_a is isomorphic to a.

To see this, notice that a is an object of $\mathcal{C}$, that id_a is a monomorphism in $\operatorname{Hom}_{\mathcal{C}}(a, a)$, and that id_a is a morphism in $\operatorname{Hom}_{\mathcal{C}}(a, a)$ such that $\mathrm{id}_a = \mathrm{id}_a \circ \mathrm{id}_a$. Furthermore, if x is an object of $\mathcal{C}$ and if $e' \in \operatorname{Hom}_{\mathcal{C}}(a, x)$ and $m' \in \operatorname{Hom}_{\mathcal{C}}(x, a)$ are morphisms such that m' is a monomorphism and $\mathrm{id}_a = m' \circ e'$, then we must show there exists a unique morphism $u : a \to x$ such that $\mathrm{id}_a = m' \circ u$. Taking $u = e'$, we have $m' \circ u = m' \circ e' = \mathrm{id}_a$, as required. The uniqueness of u follows from the fact that m' is a monomorphism. In fact, if v is any morphism in $\operatorname{Hom}_{\mathcal{C}}(a, x)$ such that $\mathrm{id}_a = m' \circ v$, then $m' \circ v = m' \circ e'$, and as a consequence, $v = e'$. Thus $u = e'$ is the unique such morphism.

The key observation in the example above is that, since the morphism itself is already a monomorphism, the image factorization does not need to factor out any additional structure. Thus, for monomorphisms, the image is essentially the domain itself. This general result will be a consequence of the next proposition.

Proposition 2.3.16. Let $\mathcal{C}$ be a category, $f \in \mathrm{Hom}_{\mathcal{C}}(a,b)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(b,c)$ be morphisms of $\mathcal{C}$. If the image of $g \circ f$ exists in $\mathcal{C}$ and g is a monomorphism, then the image of f also exists in $\mathcal{C}$ and is isomorphic to the image of $g \circ f$.

Proof. Let $\mathrm{im}(g \circ f)$ be the image of $g \circ f$, let $e \in \mathrm{Hom}_{\mathcal{C}}(a, \mathrm{im}(g \circ f))$ be a morphism, and let $m \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{im}(g \circ f), c)$ be a monomorphism such that $(g \circ f) = m \circ e$. We will show that $\mathrm{im}(g \circ f)$ also serves as the image of f.

By the universal property of $\mathrm{im}(g \circ f)$ applied to the factorization $g \circ f = g \circ f$ (where we view this as a factorization through b with $e' = f$ and $m' = g$), there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{im}(g \circ f), b)$ such that $g \circ u = m$. Moreover, since m is a monomorphism, u must also be a monomorphism. Furthermore, we have

$$g \circ f = m \circ e = (g \circ u) \circ e = g \circ (u \circ e),$$

and since g is a monomorphism, this implies that $f = u \circ e$. In summary:

- $\mathrm{im}(g \circ f)$ is an object of $\mathcal{C}$,
- $u \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{im}(g \circ f), b)$ is a monomorphism,
- $e \in \mathrm{Hom}_{\mathcal{C}}(a, \mathrm{im}(g \circ f))$ be a morphism such that $f = u \circ e$.

To complete the proof that $\mathrm{im}(g \circ f)$ is isomorphic to $\mathrm{im}(f)$, let x be an object of $\mathcal{C}$ and $e' \in \mathrm{Hom}_{\mathcal{C}}(a,x)$ and $m' \in \mathrm{Hom}_{\mathcal{C}}(x,b)$ be morphisms such that m' is a monomorphism and $f = m' \circ e'$. We need to show there exists a unique morphism $v : \mathrm{im}(g \circ f) \to x$ such that $u = m' \circ v$.

To do that, we will use the universal property of $\mathrm{im}(g \circ f)$. First notice that x is an object of $\mathcal{C}$, that $e \in \mathrm{Hom}_{\mathcal{C}}(a,x)$ is a morphism and $(g \circ m') \in \mathrm{Hom}_{\mathcal{C}}(x,c)$ is a monomorphism, such that

$$g \circ f = g \circ (m' \circ e') = (g \circ m') \circ e'.$$

Hence, the universal property of $\mathrm{im}(g \circ f)$ implies that there exists a unique morphism $v \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{im}(g \circ f), x)$ such that $m = (g \circ m') \circ v$. Since $g \circ u = m$, it follows from this equality that

$$g \circ u = (g \circ m') \circ v = g \circ (m' \circ v).$$

The fact that g is a monomorphism implies that $u = m' \circ v$, as we wanted. $\square$

In the next example, we address the opposite situation: we compute the image of the zero morphism in an abelian category.

Example 2.3.17. Let $\mathcal{A}$ be an abelian category and let a, b be objects of $\mathcal{A}$. The image of the zero morphism $0 \in \mathrm{Hom}_{\mathcal{A}}(a,b)$ is 0. To justify this claim, notice that:

- 0 is an object of $\mathcal{A}$;
- $\mathrm{id}_0 : 0 \to b$ is a monomorphism (the unique morphism from 0 to b);
- $0 : a \to 0$ is a morphism of $\mathcal{A}$ such that $0 = \mathrm{id}_0 \circ 0$ (where the left side is the zero morphism $a \to b$ and the composition on the right equals the zero morphism);
- if x is an object of $\mathcal{A}$ and if $e' \in \mathrm{Hom}_{\mathcal{A}}(a,x)$ and $m' \in \mathrm{Hom}_{\mathcal{A}}(x,b)$ are morphisms such that m' is a monomorphism and $0 = m' \circ e'$, then the unique morphism $u : 0 \to x$ (which exists since 0 is the zero object) satisfies $\mathrm{id}_0 = m' \circ u$.

These examples show cases where images exist. However, in general categories, images may fail to exist. In the next example, we will present a case where the image does not exist.

Example 2.3.18. Consider a small category $\mathcal{C}$ with two objects, $\mathrm{Obj}(\mathcal{C}) = \{a, b\}$, morphisms given by

$$\mathrm{Hom}_{\mathcal{C}}(a,a) = \mathbb{Z}_2 = \left\{\overline{0}, \overline{1}\right\}, \quad \mathrm{Hom}_{\mathcal{C}}(a,b) = \{f\},$$
$$\mathrm{Hom}_{\mathcal{C}}(b,b) = \mathbb{N} = \{0,1,2,\dots\}, \quad \mathrm{Hom}_{\mathcal{C}}(b,a) = \emptyset,$$

and composition given by

$$\overline{0} \circ \overline{0} = \overline{1} \circ \overline{1} = \overline{0}, \quad \overline{0} \circ \overline{1} = \overline{1} \circ \overline{0} = \overline{1}, \qquad f \circ \overline{0} = f \circ \overline{1} = f,$$
$$n \circ m = n + m \quad \text{and} \quad n \circ f = f \quad \text{for all } n, m \in \mathbb{N}.$$

In this case, the image of f does not exist.

To justify this claim, first notice that f is not a monomorphism, since $\overline{0} \neq \overline{1}$ and $f \circ \overline{0} = f \circ \overline{1}$. Hence, the only object of $\mathcal{C}$ for which there could exist a monomorphism into b is b itself. In fact, for all $m \in \mathrm{Hom}_{\mathcal{C}}(b,b)$, we have $m \circ n_1 = m \circ n_2$ if and only if $m + n_1 = m + n_2$ in $\mathbb{N}$. Since $m + n_1 = m + n_2$ in $\mathbb{N}$ if and only if $n_1 = n_2$, we see that every morphism in $\mathrm{Hom}_{\mathcal{C}}(b,b)$ is in fact a monomorphism. Moreover, $m \circ f = f$ for all $m \in \mathrm{Hom}_{\mathcal{C}}(b,b)$, so f factors through b via any $m \in \mathbb{N}$. However, since $\mathbb{N}$ has no maximal element and for any $m \in \mathbb{N}$ there exist $n, n' \in \mathbb{N}$ with $n > m$ such that we cannot write $m = n + n'$, there is no monomorphism $b \to b$ through which all factorizations pass. This means that the universal property cannot be satisfied, so the image of f does not exist in $\mathcal{C}$.

Having explored examples and non-examples, we close this section with the case of abelian categories, where images always exist and can be described using kernels and cokernels. The following proposition provides two equivalent characterizations of the image in this setting.

Proposition 2.3.19. Let $\mathcal{A}$ be an abelian category and let $f \in \mathrm{Hom}_{\mathcal{A}}(a, b)$ be a morphism of $\mathcal{A}$. The image of f is (isomorphic to) the cokernel of the kernel of f.

Proof. To show that $\mathrm{coker}(\ker(f))$ is isomorphic to $\mathrm{im}(f)$, let $k : \ker(f) \to a$ be the kernel of f and $e : a \to \mathrm{coker}(k)$ be the cokernel of k. From the definition of kernel (Definition 1.5.6), $f \circ k = 0$. Then, from the universal property of the cokernel (Definition 1.5.14), there exists a unique morphism $m : \mathrm{coker}(k) \to b$ such that $f = m \circ e$. Now, notice that:

- $\mathrm{coker}(k)$ is an object of $\mathcal{A}$.
- $m \in \mathrm{Hom}_{\mathcal{A}}(\mathrm{coker}(k), b)$ is a monomorphism.
- $e \in \mathrm{Hom}_{\mathcal{A}}(a, \mathrm{coker}(k))$ is a morphism that satisfies $f = m \circ e$.
- If x is an object of $\mathcal{A}$, $e' \in \mathrm{Hom}_{\mathcal{A}}(a, x)$ is a morphism, and $m' \in \mathrm{Hom}_{\mathcal{A}}(x, b)$ is a monomorphism such that $f = m' \circ e'$, then we can construct a (unique) morphism $u : \mathrm{coker}(k) \to x$ such that $m = m' \circ u$. In fact, from the definition of kernel (Definition 1.5.6), we have that

  $$0 = f \circ k = (m' \circ e') \circ k = m' \circ (e' \circ k).$$

  Since m' is a monomorphism (by hypothesis), this implies that $e' \circ k = 0$. Now, from the definition of cokernel (Definition 1.5.14), there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{A}}(\mathrm{coker}(k), x)$ such that $e' = u \circ e$. Thus,

  $$m \circ e = f = m' \circ e' = m' \circ (u \circ e) = (m' \circ u) \circ e.$$

  Since e is an epimorphism (see Example 1.7.9), this equation implies that $m = m' \circ u$.

  To finish the proof that $\mathrm{coker}(k)$ is isomorphic to $\mathrm{im}(f)$, we will use the fact that m' is a monomorphism to show that this morphism u is the unique morphism in $\mathrm{Hom}_{\mathcal{A}}(\mathrm{coker}(k), x)$ such that $m = m' \circ u$. In fact, if $u' \in \mathrm{Hom}_{\mathcal{A}}(\mathrm{coker}(k), x)$ is a morphism such that $m' \circ u' = m$, we obtain that $m' \circ u' = m = m' \circ u$, and as a consequence, that $u' = u$.

This shows that the cokernel of the kernel of f is isomorphic to its image. □

To close this section, we make a remark that follows from the proof of the previous result and will be used in the subsequent sections.

Remark 2.3.20. Let $\mathcal{A}$ be an abelian category and $f \in \mathrm{Hom}_{\mathcal{A}}(a, b)$ be a morphism in $\mathcal{A}$. It follows from the proof of Proposition 2.3.19 that the morphism e in the factorization $f = m \circ e$ given in Definition 2.3.13 is an epimorphism. Moreover, the decomposition $f = m \circ e$ is unique up to composition with isomorphisms.

2.3.4. **Exact Sequences.** Exact sequences capture the idea that the output of one morphism is precisely the input that the next morphism kills. In this section, we define exact sequences in abstract abelian categories and explore their basic properties through examples.

Definition 2.3.21 (exact sequence)**.** Given an abelian category $\mathcal{A}$, a sequence $a \xrightarrow{f} b \xrightarrow{g} c$ of two morphisms and three objects of $\mathcal{A}$ is said to be *exact at* b when $\mathrm{im}(f) = \ker(g)$. Similarly, a sequence $0 \xrightarrow{0} a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{0} 0$ of four morphisms and five objects of $\mathcal{A}$ is said to be a *short exact sequence* when it is exact at a, b and c.

Notice that the exactness of the sequence $0 \to a \to b$ at a is an abstraction of injectiveness and the exactness of the sequence $b \to c \to 0$ at c is abstraction of surjectiveness (see Proposition 2.3.27 for the formal statements). To make these concepts more concrete, we examine several examples.

Example 2.3.22. Consider the abelian category of abelian groups (see Example 1.8.3). Let $\mathbb{Z}$ denote the abelian group of integers under addition (see Example A.2), and let $\mathbb{Z}/2\mathbb{Z}$ denote the quotient group of integers modulo 2 (see Example A.21). Consider the sequence of abelian groups

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

where the first non-zero morphism is multiplication by 2 (sending $n \mapsto 2n$) and the second is the quotient map (sending $n \mapsto \overline{n}$, the residue class of n modulo 2). We verify that this is a short exact sequence by checking exactness at each object:

- The image of the zero morphism $0 \to \mathbb{Z}$ is $\{0\}$. The kernel of multiplication by 2 is $\{n \in \mathbb{Z} \mid 2n = 0\} = \{0\}$. Since $\{0\} = \{0\}$, the sequence is exact at the first $\mathbb{Z}$.
- The image of multiplication by 2 is $\{2n \mid n \in \mathbb{Z}\} = 2\mathbb{Z}$, the set of even integers. The kernel of the quotient map is $\{n \in \mathbb{Z} \mid \overline{n} = \overline{0}\} = 2\mathbb{Z}$, the set of all integers that are multiples of 2. Since $2\mathbb{Z} = 2\mathbb{Z}$, the sequence is exact at the second $\mathbb{Z}$.
- The image of the quotient map is $\mathbb{Z}/2\mathbb{Z}$, since it is surjective. The kernel of the zero morphism $\mathbb{Z}/2\mathbb{Z} \to 0$ is all of $\mathbb{Z}/2\mathbb{Z}$. Since $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$, the sequence is also exact at $\mathbb{Z}/2\mathbb{Z}$.

The concrete example above illustrates the key features of short exact sequences in a familiar setting. However, the definition applies equally well to abstract abelian categories, and we can construct exact sequences from any object using only categorical operations. The next example shows the simplest possible short exact sequence that exists in an abelian category.

Example 2.3.23. Let $\mathcal{A}$ be any abelian category and a be an object of $\mathcal{A}$. The sequence

$$0 \xrightarrow{0} a \xrightarrow{\mathrm{id}_a} a \xrightarrow{0} 0 \xrightarrow{0} 0$$

is a short exact sequence. To verify this, we check exactness at each subsequence:

- The sequence $0 \xrightarrow{0} a \xrightarrow{\mathrm{id}_a} a$ is exact at a because the image of the morphism 0 is 0 (see Example 2.3.17) and the kernel of the identity morphism is 0.
- The sequence $a \xrightarrow{\mathrm{id}_a} a \xrightarrow{0} 0$ is exact at a because the image of the identity morphism id_a is a (see Proposition 2.3.16) and the kernel of the 0 morphism is a (see Example 1.5.2).
- The sequence $a \xrightarrow{0} 0 \xrightarrow{0} 0$ is exact at 0 because the image of the 0 morphism is 0 (see Example 2.3.17) and the kernel of the 0 morphism is 0 (see Example 1.5.2).

While this example is somewhat degenerate, it establishes that exact sequences exist and that the definition is consistent with our intuition. More interesting examples arise from the fundamental constructions involving products and kernels. The next example shows how products in abelian categories induce short exact sequences.

Example 2.3.24. Let $\mathcal{A}$ be an abelian category, a and b be two objects of $\mathcal{A}$. We will construct a short exact sequence

$$0 \to a \to a \times b \to b \to 0.$$

To do that, begin by recalling from Definition 1.2.9 that there exist morphisms $p_a \in \mathrm{Hom}_{\mathcal{A}}(a \times b, a)$ and $p_b \in \mathrm{Hom}_{\mathcal{A}}(a \times b, b)$ such that: if x is an object of $\mathcal{A}$ and if $f_a \in \mathrm{Hom}_{\mathcal{A}}(x, a)$ and $f_b \in \mathrm{Hom}_{\mathcal{A}}(x, b)$ are morphisms of $\mathcal{A}$, then there exists a unique morphism $f \in \mathrm{Hom}_{\mathcal{A}}(x, a \times b)$ such that $p_a \circ f = f_a$ and $p_b \circ f = f_b$. In particular, if we choose $x = a$, $f_a = \mathrm{id}_a$ and $f_b = 0$, there exists a unique morphism $i_a \in \mathrm{Hom}_{\mathcal{A}}(a, a \times b)$ such that $p_a \circ i_a = \mathrm{id}_a$ and $p_b \circ i_a = 0$. Similarly, if we choose $x = b$, $f_a = 0$ and $f_b = \mathrm{id}_b$, there exists a unique morphism $i_b \in \mathrm{Hom}_{\mathcal{A}}(b, a \times b)$ such that $p_a \circ i_b = 0$ and $p_b \circ i_b = \mathrm{id}_b$. We will use these morphisms and show that the sequence

$$0 \to a \xrightarrow{i_a} a \times b \xrightarrow{p_b} b \to 0 \tag{2.3.1}$$

is exact.

To show that this sequence is exact at a, we will verify that, if x is an object of $\mathcal{A}$ and $f \in \mathrm{Hom}_{\mathcal{A}}(x, a)$ is a morphism such that $i_a \circ f = 0$, then $f = 0$. This will imply that $\ker(a) = 0$ (see Definition 1.5.6). In fact, if $i_a \circ f = 0$, then

$$f = \mathrm{id}_a \circ f = (p_a \circ i_a) \circ f = p_a \circ (i_a \circ f) = p_a \circ 0 = 0.$$

To show that the sequence (2.3.1) is exact at $a \times b$, by definition, we must show that $\mathrm{im}(i_a) = \ker(p_b)$. Since i_a is a monomorphism, this is equivalent to showing that (a, i_a) is the kernel-pair of p_b (see Proposition 2.3.16). So, we will verify that, if x is an object of $\mathcal{A}$ and $f \in \mathrm{Hom}_{\mathcal{A}}(x, a \times b)$ is a morphism such that $p_b \circ f = 0$, then there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{A}}(x, a)$ such that $i_a \circ u = f$. In fact, if we choose $u = p_a \circ f$, then

$$i_a \circ u = i_a \circ (p_a \circ f) = (i_a \circ p_a) \circ f = \mathrm{id}_a \circ f = f.$$

The uniqueness of u follows from the fact that i_a is a monomorphism (which was proved in the previous paragraph).

Finally, to show that the sequence (2.3.1) is exact at b, we will verify that (b, p_b) is the cokernel of i_a. Since (a, i_a) is the kernel of p_b, this will imply that $\mathrm{im}(p_b) = b$ (see Proposition 2.3.19). To verify that (b, p_b) is the cokernel of i_a, we must check that, if x is an object of $\mathcal{A}$ and $f \in \mathrm{Hom}_{\mathcal{A}}(a \times b, x)$ is a morphism such that $f \circ i_a = 0$, then there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{A}}(b, x)$ such that $u \circ p_b = f$. In fact, if we choose $u = f \circ i_b$, since $f \circ i_a = 0$, we have

$$f = f \circ \mathrm{id}_{a\times b} = f \circ (i_a \circ p_a + i_b \circ p_b) = f \circ (i_a \circ p_a) + f \circ (i_b \circ p_b) = u \circ p_b.$$

The uniqueness of u follows from the fact that $p_b \circ i_b = \mathrm{id}_b$.

The next example shows how kernels in abelian categories induce short exact sequences.

Example 2.3.25. Let $\mathcal{A}$ be an abelian category, a, b be two objects of $\mathcal{A}$, and $f \in \mathrm{Hom}_{\mathcal{A}}(a, b)$ be a morphism. We can construct a short exact sequence

$$0 \to \ker(f) \to a \to \mathrm{im}(f) \to 0.$$

To justify this claim, we will construct each subsequence of this short exact sequence and explain why it is exact.

- First, denote by $(\ker(f), k)$ the kernel-pair of the morphism f (see Definition 1.5.6). By Example 1.7.4, we know that k is a monomorphism, and hence, we know that its kernel is 0. Since the image of the 0 morphism is also 0 (see Example 2.3.17), we see that the sequence $0 \xrightarrow{0} \ker(f) \xrightarrow{k} a$ is exact at $\ker(f)$.
- Next, recall from Definition 2.3.13 that there exist $e \in \mathrm{Hom}_{\mathcal{A}}(a, \mathrm{im}(f))$ and $m : \mathrm{Hom}_{\mathcal{A}}(\mathrm{im}(f), b)$ such that m is monic and $f = m \circ e$. Since m is monic, the kernel of e is equal to the kernel of f (as $f \circ k = 0$ if and only if $e \circ k = 0$). Since k is a monomorphism, its image is also equal to $\ker(f)$ (see Proposition 2.3.16). Hence, the sequence $\ker(f) \xrightarrow{k} a \xrightarrow{e} \mathrm{im}(f)$ is exact at a.
- Finally, recall from Proposition 2.3.16 that the image of e is equal to the image of f. Since the kernel of the morphism $0 : \mathrm{im}(f) \to 0$ is $\mathrm{im}(f)$, it follows that the sequence $a \xrightarrow{e} \mathrm{im}(f) \xrightarrow{0} 0$ is exact at $\mathrm{im}(f)$.

These examples illustrate how exact sequences arise naturally from the basic morphisms in abelian categories. However, exactness is a strong condition, and most sequences are not exact. To see an evidence of this, consider the following example from linear algebra.

Example 2.3.26. Consider the sequence $\mathbb{R} \xrightarrow{T} \mathbb{R}^2 \xrightarrow{S} \mathbb{R}^3$ in the category of real vector spaces, where $T(x) = (x, 0)$ and $S(x, y) = (x, y, 0)$. This sequence is *not exact* at $\mathbb{R}^2$, since the image of T is $\mathbb{R} \times \{0\}$ and S is injective, then

$$\mathrm{im}(T) = \mathbb{R} \times \{0\} \neq \{(0, 0)\} = \ker(S).$$

We will close this section with a result that formalizes the idea that the exactness of the sequence $0 \to a \to b$ at a is an abstraction of injectivity, and the exactness of the sequence $b \to c \to 0$ at c is an abstraction of surjectivity.

Proposition 2.3.27. Let $\mathcal{A}$ be an abelian category and a, b be two of its objects.

(a) If $f \in \operatorname{Hom}_{\mathcal{A}}(a, b)$ is such that the sequence $0 \to a \xrightarrow{f} b$ is exact at a, then f is a monomorphism.

(b) If $f \in \operatorname{Hom}_{\mathcal{A}}(a, b)$ is such that the sequence $a \xrightarrow{f} b \to 0$ is exact at b, then f is an epimorphism.

Proof. We will prove each part separately.

(a) Assume the sequence $0 \to a \xrightarrow{f} b$ is exact at a. By definition of exactness, this means that $\operatorname{im}(0 \to a) = \ker(f)$. Since the image of the zero morphism $0 \to a$ is 0 (see Example 2.3.17), we have $\ker(f) = 0$.

To show that f is a monomorphism, let x be an object of $\mathcal{A}$ and let $g, h \in \operatorname{Hom}_{\mathcal{A}}(x, a)$ be morphisms such that $f \circ g = f \circ h$. We need to show that $g = h$. Since $f \circ g = f \circ h$, then $f \circ (g - h) = 0$, which implies that $g - h$ factors through $\ker(f)$. Since $\ker(f) = 0$, this means that there exists a unique morphism $u : x \to 0$ (the zero morphism) such that $g - h = k \circ u$, where $k : 0 \to a$ is the kernel morphism (see Definition 1.5.6). Since $k \circ u = 0$, so is $g - h = 0$. This implies that $g = h$ and proves that f is a monomorphism.

(b) Assume that the sequence $a \xrightarrow{f} b \to 0$ is exact at b. By definition of exactness, this means that $\operatorname{im}(f) = \ker(b \to 0)$. Since the kernel of the zero morphism $b \to 0$ is all of b (see Example 1.5.2), we have $\operatorname{im}(f) = b$. Now, recall from Remark 2.3.20 that there exist a unique (up to composition with isomorphisms) epimorphism $e \in \operatorname{Hom}_{\mathcal{A}}(a, \operatorname{im}(f))$ and a unique (up to composition with isomorphisms) monomorphism $m \in \operatorname{Hom}_{\mathcal{A}}(\operatorname{im}(f), b)$ such that $f = m \circ e$. Since $\operatorname{im}(f) = b$, we can choose $f = \operatorname{id}_b \circ f$. This implies that f is an epimorphism. □

2.4. Exact Functors

When working with categories, we are naturally interested in functors that preserve some of their structure. Functors that preserve limits and colimits are called *exact functors*. They play central roles in homological algebra, algebraic geometry, and category theory. In this section, we will define this notion precisely and illustrate it with examples. We begin with the abstract definition of exact functors.

Definition 2.4.1 (exact functors)**.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ between them is said to be: *left-exact* when all finite limits exist in $\mathcal{C}$ and F preserves these finite limits; *right-exact* when all finite colimits exist in $\mathcal{C}$ and F preserves these finite colimits, and *exact* when all finite limits and colimits exist in $\mathcal{C}$ and F preserves these finite limits and finite colimits.

We will expand a little more on the definition above. First, consider the definition of a left-exact functor. Recall that limits are defined for diagrams in categories. So, given two categories $\mathcal{I}$ and $\mathcal{C}$, a diagram D is a functor $D : \mathcal{I} \to \mathcal{C}$. The limit of this diagram is a pair $(c, \{\phi_i\}_i)$, where c is an object of $\mathcal{C}$, ϕ_i is a morphism in $\mathrm{Hom}_{\mathcal{C}}(c, D(i))$ for every object i of $\mathcal{I}$, and the pair $(c, \{\phi_i\}_i)$ satisfies the universal property of limits (see Definition 2.3.1). Moreover, this limit is said to be finite, when the category $\mathcal{I}$ has finitely-many objects and finitely-many morphisms, that is, $\mathrm{Obj}(\mathcal{I})$ and $\mathrm{Mor}(\mathcal{C})$ are finite sets.

Next, recall that for every category $\mathcal{D}$ and every functor $F : \mathcal{C} \to \mathcal{D}$, we can define a composition functor $F \circ D : \mathcal{I} \to \mathcal{D}$ (see Proposition 2.1.6). Notice that $F \circ D$ is a diagram in $\mathcal{D}$. Hence, we can also define the limit of this diagram $F \circ D$ in $\mathcal{D}$. By definition, F is left-exact when $(F(c), \{F(\phi_i)\}_i)$ is the limit of the diagram $F \circ D$ for every diagram $D : \mathcal{I} \to \mathcal{C}$ whose limit in $\mathcal{C}$ is $(c, \{\phi_i\}_i)$.

Similarly, a functor is said to be right-exact when the pair $(F(c), \{F(\phi_i)\}_i)$ is the colimit of the diagram $F \circ D$ for every diagram $D : \mathcal{I} \to \mathcal{C}$ for which $(c, \{\phi_i\}_i)$ is the colimit in $\mathcal{C}$. Notice that these definitions capture different ways in which a functor can interact with limits and colimits. To understand these distinctions, we begin with the simplest case where exactness is automatic.

Example 2.4.2. For every category $\mathcal{C}$ in which all finite limits and colimits exist, the identity functor $\mathrm{Id}_{\mathcal{C}}$ (see Example 2.1.2) is exact. To justify this claim, suppose $\mathcal{I}$ is a finite category and $D : \mathcal{I} \to \mathcal{C}$ is a diagram in $\mathcal{C}$.

If the limit of a diagram D exists, it will be a pair $(c, \{\phi_i\}_i)$, where c is an object of $\mathcal{C}$, ϕ_i is a morphism in $\mathrm{Hom}_{\mathcal{C}}(c, D(i))$ for every object i of $\mathcal{I}$, and the universal property of limits is satisfied by $(c, \{\phi_i\}_i)$. When we apply the functor $\mathrm{Id}_{\mathcal{C}}$ to this pair, we obtain the pair $(\mathrm{Id}_{\mathcal{C}}(c), \{\mathrm{Id}_{\mathcal{C}}(\phi_i)\}_i)$, where $\mathrm{Id}_{\mathcal{C}}(c) = c$ and $\mathrm{Id}_{\mathcal{C}}(\phi_i) = \phi_i$ for all $i \in \mathrm{Obj}(\mathcal{I})$. Since $\mathrm{Id}_{\mathcal{C}} \circ D = D$, the limit of the diagram $\mathrm{Id}_{\mathcal{C}} \circ D$ is exactly the pair $(\mathrm{Id}_{\mathcal{C}}(c), \{\mathrm{Id}_{\mathcal{C}}(\phi_i)\}_i)$. This shows that $\mathrm{Id}_{\mathcal{C}}$ preserves limits, that is, it is left-exact.

If the colimit of the diagram D also exists, it will be a pair $(c, \{\psi_i\}_i)$, where c is an object of $\mathcal{C}$, ψ_i is a morphism in $\mathrm{Hom}_{\mathcal{C}}(D(i), c)$ for every object i of $\mathcal{I}$, and the universal property of colimits is satisfied by this pair. When we apply the functor $\mathrm{Id}_{\mathcal{C}}$, we obtain the pair $(\mathrm{Id}_{\mathcal{C}}(c), \{\mathrm{Id}_{\mathcal{C}}(\psi_i)\}_i) = (c, \{\psi_i\}_i)$. Hence, the colimit of the diagram $\mathrm{Id}_{\mathcal{C}} \circ D = D$ is exactly the pair $(\mathrm{Id}_{\mathcal{C}}(c), \{\mathrm{Id}_{\mathcal{C}}(\psi_i)\}_i)$. This shows that $\mathrm{Id}_{\mathcal{C}}$ also preserves colimits, that is, it is also right-exact.

Identity functors provide no obstruction to exactness because they preserve all structures of the category. More interesting examples arise from Hom-functors, which behave differently with respect to exactness.

Example 2.4.3. Let $\mathcal{C}$ be a category in which all finite limits and colimits exist, let x be a fixed object of $\mathcal{C}$, and denote by $\mathcal{D}$ the category of sets (see Example 1.1.3). The covariant Hom-functor $\mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathcal{D}$ (see Example 2.1.4) is left-exact but generally not right-exact.

To justify the claim that $\mathrm{Hom}_{\mathcal{C}}(x, -)$ is left-exact, we will show that it preserves finite limits. To do that, let $\mathcal{I}$ be a finite category, let $D : \mathcal{I} \to \mathcal{C}$ be a diagram in $\mathcal{C}$, and denote by $(c, \{\psi_i\}_i)$ its limit. We want to show that the pair $(\mathrm{Hom}_{\mathcal{C}}(x, c), \{\mathrm{Hom}_{\mathcal{C}}(x, \psi_i)\}_i)$ is the limit of the diagram $\mathrm{Hom}_{\mathcal{C}}(x, -) \circ D$.

To do that, let C be a set and $f_i : C \to \mathrm{Hom}_{\mathcal{C}}(x, D(i))$ be a function for each $i \in \mathrm{Obj}(\mathcal{I})$. We must construct a function $u : C \to \mathrm{Hom}_{\mathcal{C}}(x, c)$ such that $\psi_i \circ u = f_i$ for all $i \in \mathrm{Obj}(\mathcal{I})$. To do that, first notice that, for each element $\star \in C$, we have a family of functions $\{f_i(\star) : x \to D(i)\}_i$. Since $(c, \{\psi_i\}_i)$ is the limit of the diagram D, there exists a unique morphism $u_\star : x \to c$ such that $\psi_i \circ u_\star = f_i(\star)$. Hence, we can define a function $u : C \to \mathrm{Hom}_{\mathcal{C}}(x, c)$ by setting $u(\star) = u_\star$. The uniqueness of the function u follows from the uniquenesses of each one of the morphisms $u_\star$.

To show that $\mathrm{Hom}_{\mathcal{C}}(x, -)$ is not necessarily right-exact, consider the case where $\mathcal{C}$ is abelian. Then, recall hat there exists a zero object 0 in $\mathcal{C}$ and that $\mathrm{Hom}_{\mathcal{C}}(x, 0)$ contains exactly one morphism (since 0 is also terminal). However, the set with one element is not an initial object in the category of sets (see

Example 1.2.6). This means that the functor $\mathrm{Hom}_{\mathcal{C}}(x, -)$ does not preserve initial objects, which are colimits of empty diagrams (see Example 2.3.9).

Hom-functors illustrate the case of functors that are left-exact and not right-exact. In the next example, we will show a functor that is right-exact and not left-exact.

Example 2.4.4. Let **Sets** be the category of sets (see Example 1.1.3), and recall from Example 2.3.2 and Example 2.3.8 that all finite limits and colimits exist in **Sets**. Then, choose a set S with more than one element and define a functor $F : \mathbf{Sets} \to \mathbf{Sets}$ by assigning:

- the set $F(X) := S \times X$ to each set X,
- the function $F(f) := (\mathrm{id}_S \times f)$ to each function $f : X \to Y$.

We will show that this functor is right-exact but not left-exact.

To show that F preserves finite colimits, let $\mathcal{J}$ be a finite category and $D : \mathcal{J} \to \mathbf{Sets}$ be a diagram in **Sets**. Recall from Example 2.3.8 that the colimit of D is the pair $(C, \{\iota_i\}_i)$, where:

- C is the quotient of the set $\bigsqcup_i D(i)$ by the equivalence relation $\sim$ generated by $d_i \sim d_j$ if $d_i \in D(i)$, $d_j \in D(j)$, $i, j \in \mathrm{Obj}(\mathcal{J})$, and there exists a morphism $f \in \mathrm{Hom}_{\mathcal{J}}(i, j)$ such that $D(f)(d_i) = d_j$,
- for each $i \in \mathrm{Obj}(\mathcal{J})$, the function $\iota_i : D(i) \to C$ identifies an element $d_i \in D(i)$ with its corresponding equivalence class inside $\bigsqcup_i D(i)/\sim$.

Hence, $F(\mathrm{colim}\, D) = (C \times S,\ \{\iota_i \times \mathrm{id}_S\}_i)$.

Now, consider the diagram $(F \circ D) : \mathcal{J} \to \mathbf{Sets}$. We want to show that the colimit of $F \circ D$ is the same as $F(\mathrm{colim}\, D)$. To do that, recall from Example 2.3.8 again that $\mathrm{colim}(F \circ D)$ is the pair $(C', \{\phi_i\}_i)$ where:

- C' is the quotient of the set $\bigsqcup_i (D(i) \times S)$ by the equivalence relation generated by $(d_i, s) \approx (d_j, s')$ if $d_i \in D(i)$, $d_j \in D(j)$, $s, s' \in S$ and there exists a morphism $f \in \mathrm{Hom}_{\mathcal{J}}(i, j)$ such that $(D(f) \times \mathrm{id}_S)\,(d_i, s) = (d_j, s')$,
- for each $i \in \mathrm{Obj}(\mathcal{J})$, the function $\phi_i : D(i) \times S \to C'$ identifies an element $(d_i, s) \in D(i) \times S$ with its equivalence class inside $\bigsqcup_i (D(i) \times S)/\approx$.

Using these descriptions it becomes easy to see that $F(C) = C'$ and $F(\iota_i) = \phi_i$ for all $i \in \mathrm{Obj}(\mathcal{J})$. This shows that F preserves finite colimits.

Now, to show that F does not preserve finite limits, we show that it fails to preserve terminal objects. The terminal object in **Sets** is a set with one element However, $F(\{\bullet\}) = S \times \{\bullet\}$ is not a terminal object in **Sets**, since S has more than one element. Therefore, F does not preserve terminal objects, which are limits of empty diagrams (see Example 2.3.3), and hence F is not left-exact.

We have now seen functors that are fully exact, only left-exact and only right-exact. To complete the picture, we will present an example of a functor that fails to be exact in any sense.

Example 2.4.5. Consider the category of sets from Example 1.1.3. Recall that all finite limits and colimits exist in this category (see Example 2.3.2 and Example 2.3.8). Then, let $F : \mathbf{Sets} \to \mathbf{Sets}$ be the functor defined by assigning:

- the set $\{0, 1\}$ to every set in $\mathrm{Obj}(\mathbf{Sets})$,
- the identity function of the set $\{0, 1\}$ to every function in $\mathrm{Mor}(\mathbf{Sets})$.

This functor is neither left-exact nor right-exact. It is not left-exact because it does not preserve terminal objects in **Sets**, which are the sets with one element (see Example 1.2.6). It is not right-exact because it does not preserve the initial object in **Sets**, which is the empty set (see Example 1.2.6). Since a functor is left-exact if it preserves all finite limits (including terminal objects, see Example 2.3.3) and right-exact if it preserves all finite colimits (including initial objects, see Example 2.3.9), F is neither.

We close this section with a result that characterizes exact functors on abelian categories in terms of kernels, cokernels and exact sequences.

Proposition 2.4.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(a) If all finite limits exist in $\mathcal{A}$, then: F is left-exact if and only if F is additive and $F(\ker(f)) = \ker(F(f))$ for every $f \in \mathrm{Mor}(\mathcal{A})$.

(b) If all finite colimits exist in $\mathcal{A}$, then: F is right-exact if and only if F is additive and $F(\mathrm{coker}(f)) = \mathrm{coker}(F(f))$ for every $f \in \mathrm{Mor}(\mathcal{A})$.

(c) If all finite limits and colimits exist in $\mathcal{A}$, then: F is exact if and only if F is additive and, for every exact sequence $0 \to a \to b \to c \to 0$ in $\mathcal{A}$, the sequence $0 \to F(a) \to F(b) \to F(c) \to 0$ is exact in $\mathcal{B}$.

Proof. We will prove only part (a), as he proof of part (b) is very similar to that of part (a) and part (c) follows from part (a) and part (b).

On the one hand, assume F that is left-exact. First, we will prove that F is additive. Since $\mathcal{A}$ is abelian, it has a zero object 0, which is terminal. Since F preserves finite limits and terminal objects are finite limits, F preserves terminal objects. In particular, $F(0)$ is a terminal object of $\mathcal{B}$. In an abelian category, any terminal object is also initial. Hence $F(0)$ is a zero object in $\mathcal{B}$.

Now consider any two objects a and b of $\mathcal{A}$. Their product $a \times b$ exists and is a finite limit. Since F preserves finite limits, $F(a \times b)$ is isomorphic to $F(a) \times F(b)$. In an abelian category, finite products coincide with finite coproducts. This implies that F also preserves finite biproducts. Therefore, F is additive.

The fact that F also preserves kernels follows from the fact that F preserves finite limits and the fact that kernels (or, more generally, equalizers) are finite limits.

On the other hand, assume F is additive and preserves kernels. We show F preserves all finite limits. To do that, we will show that every finite limit is isomorphic with the equalizer of certain morphisms between products in $\mathcal{A}$. This is a general construction that, in this case, will imply that F preserves finite limits, since it is additive (preserves terminal objects and finite products) and preserves kernels.

We begin by considering two finite products within $\mathcal{A}$. First, define the product $P_1 := \prod_{i \in \mathrm{Obj}(\mathcal{J})} D(i)$. Then, for each morphism $f \in \mathrm{Hom}_{\mathcal{J}}(i,j)$, denote the object $j \in \mathrm{Obj}(\mathcal{J})$ by j_f, and define the product $P_2 := \prod_{f \in \mathrm{Mor}(\mathcal{J})} D(j_f)$. The universal projections $P_1 \to D(i)$ will be denoted by π_i, while the universal projections $P_2 \to D(j_f)$ will be denoted by π_f (see Definition 1.2.9).

Now, we will construct two morphisms between P_1 and P_2. To construct the first one, notice that, for every $f \in \mathrm{Hom}_{\mathcal{J}}(i,j)$, there is a morphism $\pi_j \in \mathrm{Hom}_{\mathcal{A}}(P_1, D_j)$. Hence, by the universal property of products (see Definition 1.2.9), there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{A}}(P_1, P_2)$ such that $\pi_f \circ u = \pi_{j_f}$ for all $f \in \mathrm{Mor}(\mathcal{A})$. To construct the second one, notice that there is a morphism $(D(f) \circ \pi_{i_f}) \in \mathrm{Hom}_{\mathcal{A}}(P_1, D(j_f))$ for every morphism $f \in \mathrm{Hom}_{\mathcal{J}}(i_f, j_f)$. Hence, by the universal property of products (see Definition 1.2.9), there exists a unique morphism $v \in \mathrm{Hom}_{\mathcal{A}}(P_1, P_2)$ such that $\pi_f \circ v = D(f) \circ \pi_{i_f}$ for all $f \in \mathrm{Mor}(\mathcal{A})$.

To finish this proof, we will show that the equalizer of u and v is the limit of the diagram D. To do that, begin by denoting this equalizer by (E, e), where E is an object of $\mathcal{A}$ and $e \in \mathrm{Hom}_{\mathcal{A}}(E, P_1)$ is the universal morphism such that $u \circ e = v \circ e$ (see Definition 1.5.1). Then, notice that $(E, \{\pi_i \circ e\}_i)$ is a cone over D. In fact, if $f \in \mathrm{Hom}_{\mathcal{J}}(i, j)$, then

$$\begin{aligned} D(f) \circ (\pi_i \circ e) &= (D(f) \circ \pi_i) \circ e \\ &= (\pi_f \circ v) \circ e \\ &= \pi_f \circ (v \circ e) \\ &= \pi_f \circ (u \circ e) \\ &= (\pi_f \circ u) \circ e \\ &= \pi_j \circ e. \end{aligned}$$

This shows that $(E, \{\pi_i \circ e\}_i)$ is a cone over D. Next, given any cone $(c, \{\phi_i\}_i)$ over D, we have to construct a unique morphism $w \in \mathrm{Hom}_{\mathcal{A}}(c, E)$ such that $(\pi_i \circ e) \circ w = \phi_i$ for all $i \in \mathrm{Obj}(\mathcal{J})$. To do that, first notice that the universal property of products implies that there exists a morphism $\phi \in \mathrm{Hom}_{\mathcal{A}}(c, P_1)$ such that $\pi_i \circ \phi = \phi_i$ for all $i \in \mathrm{Obj}(\mathcal{J})$ (see Definition 1.2.9). Next, notice that the fact that $(c, \{\phi_i\}_i)$ is a cone over D implies that $u \circ \phi = v \circ \phi$. Hence, the universal property of equalizers implies that there exists a unique morphism $w \in \mathrm{Hom}_{\mathcal{A}}(c, E)$ such that $e \circ w = \phi$ (see Definition 1.5.1). As a consequence, $\pi_i \circ e \circ w = \pi_i \circ \phi = \phi_i$ for all $i \in \mathrm{Obj}(\mathcal{J})$. This completes the proof that (E, e) is the limit of the diagram D in $\mathcal{A}$. □

2.5. Adjoint Functors

The concept of adjoint functors captures one of the most fundamental relationships in category theory: a natural correspondence between morphism sets in two different categories. Adjunctions unify numerous mathematical phenomena and universal properties. In this section, we will introduce the abstract definition of adjoint functors, explore several key examples, and establish their basic properties regarding exactness.

Definition 2.5.1 (adjoint functors)**.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, and a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, we say that F and G are *adjoint functors* if there exist natural bijections

$$\mathrm{Hom}_{\mathcal{D}}(F(c), d) \cong \mathrm{Hom}_{\mathcal{C}}(c, G(d)).$$

In this case, we say that F is *left adjoint* to G, and that G is *right adjoint* to F.

The definition above captures a fundamental pattern: having a morphism from $F(c)$ to d in $\mathcal{D}$ is equivalent to having a morphism from c to $G(d)$ in $\mathcal{C}$. This correspondence preserves the relevant structure of morphism composition through the naturality condition. This translation between categories manifests in numerous mathematical contexts, as the following examples illustrate. To build intuition, we begin with the simplest cases where a functor is adjoint to itself.

Example 2.5.2. For every category $\mathcal{C}$, the identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is both left and right adjoint to itself. Indeed, in this case, we have equalities:

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{Id}_{\mathcal{C}}(c_1), c_2) = \mathrm{Hom}_{\mathcal{C}}(c_1, c_2) = \mathrm{Hom}_{\mathcal{C}}(c_1, \mathrm{Id}_{\mathcal{C}}(c_2)).$$

The naturality condition is automatically satisfied since the identity functor acts trivially on both objects and morphisms.

The identity adjunction, while simple, demonstrates the reflexive nature of adjoint relationships. More interesting examples arise when the categories involved have additional structure, particularly zero objects.

Example 2.5.3. Let $\mathcal{A}$ be an abelian category with zero object 0. The zero functor $Z : \mathcal{A} \to \mathcal{A}$, which assigns every object to 0 and every morphism to the zero morphism $\mathrm{id}_0 : 0 \to 0$, is both left and right adjoint to itself. Indeed, we have natural bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(Z(a_1), a_2) &= \mathrm{Hom}_{\mathcal{A}}(0, a_2) \\ &= \{0 : 0 \to a_2\} \\ &\cong \{0 : a_1 \to 0\} \\ &= \mathrm{Hom}_{\mathcal{A}}(a_1, 0) \\ &= \mathrm{Hom}_{\mathcal{A}}(a_1, Z(a_2)). \end{aligned}$$

The naturality of this bijection follows from the fact that composition with any morphism involving the zero object always yields the zero morphism.

The example above is a particular case of a general phenomenon: every universal construction gives rise to an adjunction. More precisely, limits and colimits correspond to adjunctions between a category and a functor category (or diagram category). While making this relationship fully rigorous requires the notion of *natural transformations*, which will be developed in subsequent sections, we can already observe that universal properties inherently exhibit the pattern of adjoint relationships.

The previous examples feature symmetric adjunctions where the same functor serves as both left and right adjoint. More commonly, however, adjoint relationships are asymmetric, with each functor playing a distinct role. A paradigmatic example of such asymmetry is the adjunction between free and forgetful functors, which we now examine.

Example 2.5.4. Let $\Bbbk$ be a field, let $\mathcal{A}$ be the category of $\Bbbk$-vector spaces, and let $\mathcal{B}$ be the category of sets. The forgetful functor $F : \mathcal{A} \to \mathcal{B}$ is defined by assigning:

- to each $\Bbbk$-vector space, its underlying set,
- to each linear transformation, its underlying function.

The fact that F is a functor follows from the observation that the composition of linear transformations is the composition of their underlying functions and the observation that the identity linear transformation is the identity function on the underlying set.

This functor F has a left adjoint $G : \mathcal{B} \to \mathcal{A}$, which is defined by assigning:

- to each set S, the vector space with basis S (the free vector space generated by S),
- to each function $f : S \to S'$, the unique linear transformation $G(f) : G(S) \to G(S')$ that extends f linearly:

$$G(f)\,(\lambda_1 s_1 + \cdots + \lambda_n s_n) = \lambda_1 f(s_1) + \cdots + \lambda_n f(s_n).$$

The fact that G is left adjoint to F, follows from the fact that there exist natural bijections

$$\mathrm{Hom}_{\mathcal{A}}(G(S), V) \cong \mathrm{Hom}_{\mathcal{B}}(S, F(V)),$$

since any function from a set S to the underlying set of a vector space V extends uniquely to a linear transformation from the vector space generated by S to V.

This adjunction between freely generated and forgetful functors exemplifies a pattern that appears throughout algebra. Another fundamental construction that leads to adjunctions is the tensor product of vector spaces, which we examine next.

Example 2.5.5. Let $\mathbb{k}$ be a field and let $\mathcal{C}$ be the category of $\mathbb{k}$-vector spaces. Recall that the tensor product of $\mathbb{k}$-vector spaces is defined as follows. Given two $\mathbb{k}$-vector spaces V and W, their tensor product is a $\mathbb{k}$-vector space $V \otimes_{\mathbb{k}} W$ endowed with a bilinear map $\iota : V \times W \to V \otimes_{\mathbb{k}} W$ satisfying the universal property: for every $\mathbb{k}$-vector space U and bilinear map $B : V \times W \to U$, there exists a unique linear transformation $T : V \otimes_{\mathbb{k}} W \to U$ such that $B = T \circ \iota$. We will see how this universal property leads to an adjunction of functors.

First, notice that the tensor product induces a functor from $\mathcal{C}$ to itself. Indeed, given a fixed $\mathbb{k}$-vector space W, we have a functor $F : \mathcal{C} \to \mathcal{C}$ that assigns:

- to each $\mathbb{k}$-vector space V, the $\mathbb{k}$-vector space $F(V) = V \otimes_{\mathbb{k}} W$,
- to each linear transformation $T : V_1 \to V_2$, the linear transformation $F(T) : V_1 \otimes_{\mathbb{k}} W \to V_2 \otimes_{\mathbb{k}} W$ given by

$$F(T)\,(v_1 \otimes w_1 + \cdots v_n \otimes w_n) = T(v_1) \otimes w_1 + \cdots + T(v_n) \otimes w_n.$$

The fact that F is a functor follows from the bilinearity of the tensor product: $F(\mathrm{id}_V) = \mathrm{id}_{V \otimes W}$ and $F(T_2 \circ T_1) = F(T_2) \circ F(T_1)$.

To construct a right adjoint to F, recall the universal property of the tensor product: for every $\mathbb{k}$-vector space U and bilinear map $B : V \times W \to U$, there exists a unique linear transformation $T : V \otimes_{\mathbb{k}} W \to U$ such that $B = T \circ \iota$. This establishes a bijection between linear transformations in $\mathrm{Hom}_{\mathcal{C}}(V \otimes_{\mathbb{k}} W, U)$ and bilinear maps $B : V \times W \to U$.

Now, observe that we can identify bilinear maps $B : V \times W \to U$ with linear transformations $\phi : V \to \mathrm{Hom}_{\mathbb{k}}(W, U)$ via the correspondence:

$$B(v, w) = (\phi(v))(w).$$

Indeed, given a bilinear map B, for each $v \in V$, the map $w \mapsto B(v, w)$ is linear in w, so it defines an element of $\mathrm{Hom}_{\mathbb{k}}(W, U)$. The assignment $v \mapsto (w \mapsto B(v, w))$ is linear in v by the bilinearity of B. Conversely, given $\phi : V \to \mathrm{Hom}_{\mathbb{k}}(W, U)$, define $B(v, w) = (\phi(v))(w)$, which is bilinear. These constructions are mutually inverse.

Combining these bijections, we obtain natural bijections

$$\mathrm{Hom}_{\mathcal{C}}(V \otimes_{\mathbb{k}} W, U) \cong \mathrm{Hom}_{\mathcal{C}}(V, \mathrm{Hom}_{\mathbb{k}}(W, U)).$$

The naturality of these bijections follows from their construction via universal properties. This means that the functor $F(V) = V \otimes_{\Bbbk} W$ is left adjoint to the functor $G(U) = \mathrm{Hom}_{\Bbbk}(W, U)$.

Having explored various examples of adjoint functors, we now turn to their fundamental interaction with exactness properties. This relationship is particularly important in homological algebra, as it provides powerful tools for transferring exact sequences between categories. The key result is that left adjoints preserve colimits while right adjoints preserve limits, and in particular, left adjoints are right-exact while right adjoints are left-exact.

Proposition 2.5.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories with finite limits and colimits, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between these categories. If F is left adjoint to G, then F is right-exact and G is left-exact.

Proof. We will prove that G is left-exact. The proof that F is right-exact is completely analogous.

To show that G is left-exact, let $\mathcal{I}$ be a finite category and $D : \mathcal{I} \to \mathcal{D}$ be a diagram in $\mathcal{D}$. Since $\mathcal{D}$ has finite limits, there exists a limit of D, which we denote by $(d, \{\pi_i\}_i)$. We will show that the pair $(G(d), \{G(\pi_i)\}_i)$ is the limit of the diagram $(G \circ D) : \mathcal{I} \to \mathcal{C}$ in $\mathcal{C}$.

To do that, first, we verify that $(G(d), \{G(\pi_i)\}_i)$ forms a cone over $G \circ D$. In fact, for every morphism $f \in \mathrm{Hom}_{\mathcal{I}}(i, j)$, we have

$$G(D(f)) \circ G(\pi_i) = G(D(f) \circ \pi_i) = G(\pi_j),$$

since $(d, \{\pi_i\}_i)$ is a cone over D and G is a functor.

Next, to verify that $(G(d), \{G(\pi_i)\}_i)$ satisfies the universal property of the limit of $G \circ D$, let c be an object of $\mathcal{C}$ and $\{\phi_i\}_i$ be a family of morphisms such that $G(D(f)) \circ \phi_i = \phi_j$ for every morphism $f \in \mathrm{Hom}_{\mathcal{I}}(i, j)$. We will show there exists a unique morphism $v \in \mathrm{Hom}_{\mathcal{C}}(c, G(d))$ such that $G(\pi_i) \circ v = \phi_i$ for all $i \in \mathrm{Obj}(\mathcal{I})$.

Since F is left adjoint to G, each morphism $\phi_i : c \to G(D(i))$ corresponds to a unique morphism $\psi_i : F(c) \to D(i)$ in $\mathcal{D}$ via the adjunction bijection

$$\mathrm{Hom}_{\mathcal{C}}(c, G(D(i))) \cong \mathrm{Hom}_{\mathcal{D}}(F(c), D(i)).$$

The naturality of these bijections and the fact that $G(D(f)) \circ \phi_i = \phi_j$ imply that $D(f) \circ \psi_i = \psi_j$ for every morphism $f \in \mathrm{Hom}_{\mathcal{I}}(i, j)$. Therefore, $\{\psi_i\}_i$ forms a cone over the diagram D. By the universal property of the limit $(d, \{\pi_i\}_i)$, there exists a unique morphism $u \in \mathrm{Hom}_{\mathcal{D}}(F(c), d)$ such that $\pi_i \circ u = \psi_i$ for all $i \in \mathrm{Obj}(\mathcal{I})$. Now, the adjunction bijection

$$\mathrm{Hom}_{\mathcal{D}}(F(c), d) \cong \mathrm{Hom}_{\mathcal{C}}(c, G(d))$$

assigns the morphism u to a unique morphism $v \in \mathrm{Hom}_{\mathcal{C}}(c, G(d))$. We claim that this v satisfies $G(\pi_i) \circ v = \phi_i$ for all $i \in \mathrm{Obj}(\mathcal{I})$. This follows from the naturality of the adjunction bijections applied to π_i and the fact that $\pi_i \circ u = \psi_i$ corresponds to ϕ_i.

The uniqueness of v follows from the uniqueness of u and the bijectivity of the adjunction. This shows that $(G(d), \{G(\pi_i)\}_i)$ is the limit of the diagram $G \circ D$ in $\mathcal{C}$, which means that G preserves finite limits, or equivalently, that G is a left-exact functor. □

Part III

Tensor categories

The central theme of this part is categories endowed with an internal multiplication, known as *monoidal categories* (or tensor categories). We formally define the tensor bifunctors and the unit objects, paying particular attention to the coherence constraints (the associator and unitor natural isomorphisms) that govern them. Finally, we refine this structure by introducing commutativity. We progress from *braided monoidal categories*, where the order of tensor factors can be exchanged via an isomorphism, to *symmetric monoidal categories*, where this exchange is involutive. These definitions provide the essential framework for studying algebra-like structures within categories. We begin this part, however, with the concepts of *natural transformations* and *equivalences of categories*, which will be used throughout the remainder of the text.

3.1. Natural Transformations

While functors relate categories, natural transformations relate functors themselves. That is, they provide a way to compare two functors in a manner that respects their underlying structures. In this section, we will define natural transformations, provide examples, and prove that natural transformations can also be composed.

Definition 3.1.1. Given categories, $\mathcal{C}$ and $\mathcal{D}$, and given functors between them, $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta : F \Rightarrow G$ between these functors consists of a family of morphisms,

$$\{\eta_X : F(X) \to G(X) \in \mathrm{Mor}(\mathcal{D}) \mid X \in \mathrm{Obj}(\mathcal{C})\},$$

satisfying the following *naturality condition*:

$$G(f) \circ \eta_X = \eta_Y \circ F(f) \quad \text{for every} \quad f \in \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

A natural transformation $\eta : F \Rightarrow G$ is said to be a *natural isomorphism* when $\eta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ is an isomorphism for all $X \in \mathrm{Obj}(\mathcal{C})$.

To illustrate the abstract definition of a natural transformation given above, we will start with the simplest example, that of the identity transformation.

Example 3.1.2. For any functor $F : \mathcal{C} \to \mathcal{D}$, the identity natural transformation $\mathrm{id}_F : F \Rightarrow F$ is given by

$$(\mathrm{id}_F)_X = \mathrm{id}_{F(X)} \quad \text{for all } X \in \mathrm{Obj}(\mathcal{C}).$$

The properties of the identity morphisms imply that id_F satisfies the naturality condition; in fact,

$$F(f) \circ \mathrm{id}_{F(X)} = F(f) = \mathrm{id}_{F(Y)} \circ F(f)$$

for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Notice that id_F is also a natural isomorphism.

To give a less trivial example of a natural transformation, we will consider a category with two objects and two functors from this category to itself.

Example 3.1.3. Let $\mathcal{C}$ be the category with two objects, $\mathrm{Obj}(\mathcal{C}) = \{A, B\}$, and three morphisms, $\mathrm{Mor}(\mathcal{C}) = \{\mathrm{id}_A, f, \mathrm{id}_B\}$, where $f : A \to B$. There exist three different functors from $\mathcal{C}$ to itself: the identity functor $\mathrm{Id}_{\mathcal{C}}$; the functor $F : \mathcal{C} \to \mathcal{C}$ defined by

$$F(A) = F(B) = A \quad \text{and} \quad F(\mathrm{id}_A) = F(f) = F(\mathrm{id}_B) = \mathrm{id}_A;$$

and the functor $G : \mathcal{C} \to \mathcal{C}$ defined by

$$G(A) = G(B) = B \quad \text{and} \quad G(\mathrm{id}_A) = G(f) = G(\mathrm{id}_B) = \mathrm{id}_B\,.$$

A natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow F$ would consist of two morphisms,

$$\eta_A : A \to A \quad \text{and} \quad \eta_B : B \to A.$$

Since there exist no morphisms in $\mathrm{Hom}_{\mathcal{C}}(B, A)$, no such natural transformation exists. Similarly, no natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow G$ exists. Now, a natural transformation $\eta : F \Rightarrow G$ consists of two morphisms,

$$\eta_A : F(A) \to G(A) \quad \text{and} \quad \eta_B : F(B) \to G(B),$$

satisfying naturality conditions. Since $F(A) = F(B) = A$, $G(A) = G(B) = B$ and $\mathrm{Hom}_{\mathcal{C}}(A, B) = \{f\}$, then $\eta_A = \eta_B = f$. In this case, the naturality conditions are satisfied, since $F(\phi) = \mathrm{id}_A$ and $G(\phi) = \mathrm{id}_B$ for all $\phi \in \{\mathrm{id}_A, f, \mathrm{id}_B\}$. In fact,

$$F(\phi) \circ \eta_A = \mathrm{id}_A \circ f = f = f \circ \mathrm{id}_B = \eta_B \circ G(\phi),$$

for all $\phi \in \{\mathrm{id}_A, f, \mathrm{id}_B\}$. This shows an example of a natural transformation different from the identity one. Moreover, since f is not an isomorphism in $\mathcal{D}$, this natural transformation is also not a natural isomorphism.

In the next section, we will use natural transformations to define equivalences of categories. Then, we will provide other examples of natural transformations. Before that, we will state and prove a technical result regarding natural transformations that will also be used in the next section.

Proposition 3.1.4. Let $\mathcal{C}, \mathcal{D}$ be categories, let $E, F, G, H : \mathcal{C} \to \mathcal{D}$ be functors, and let $\zeta : E \Rightarrow F$, $\eta : F \Rightarrow G$, $\theta : G \Rightarrow H$ be natural transformations.

(a) The family $(\theta \circ \eta) : F \Rightarrow H$, defined by

$$(\theta \circ \eta)_X = \theta_X \circ \eta_X \quad \text{for each } X \in \mathrm{Obj}(\mathcal{C}),$$

is a natural transformation.

(b) The natural transformations $(\theta \circ \eta) \circ \zeta$ and $\theta \circ (\eta \circ \zeta)$ are equal.

(c) The natural transformation $\eta : F \Rightarrow G$ is a natural isomorphism if and only if there exists a natural transformation $\eta' : G \Rightarrow F$ such that

$$\eta' \circ \eta = \mathrm{id}_F \quad \text{and} \quad \eta \circ \eta' = \mathrm{id}_G .$$

Proof. (a) We need to verify that the family $\{(\theta \circ \eta)_X \mid X \in \mathrm{Obj}(\mathcal{C})\}$ satisfies the corresponding naturality conditions. To do that, notice that, for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we have

$$\begin{aligned} H(f) \circ (\theta \circ \eta)_X &= H(f) \circ (\theta_X \circ \eta_X) \\ &= (H(f) \circ \theta_X) \circ \eta_X \\ &= (\theta_Y \circ G(f)) \circ \eta_X \\ &= \theta_Y \circ (G(f) \circ \eta_X) \\ &= \theta_Y \circ (\eta_Y \circ F(f)) \\ &= (\theta_Y \circ \eta_Y) \circ F(f) \\ &= (\theta \circ \eta)_Y \circ F(f). \end{aligned}$$

This shows that $\theta \circ \eta$ satisfies the naturality conditions and thus, that it is a natural transformation.

(b) From item (a), we know that $(\theta \circ \eta) \circ \zeta$ and $\theta \circ (\eta \circ \zeta)$ are natural transformations. Using their explicit definitions, we can see that, for every object

$X \in \text{Obj}(\mathcal{C})$, we have:

$$\begin{aligned}((\theta \circ \eta) \circ \zeta)_X &= (\theta \circ \eta)_X \circ \zeta_X \\ &= (\theta_X \circ \eta_X) \circ \zeta_X \\ &= \theta_X \circ (\eta_X \circ \zeta_X) \\ &= \theta_X \circ (\eta \circ \zeta)_X \\ &= (\theta \circ (\eta \circ \zeta))_X.\end{aligned}$$

This means that $(\theta \circ \eta) \circ \zeta = \theta \circ (\eta \circ \zeta)$, as we wanted to show.

(c) Recall from Definition 3.1.1 that the natural transformation $\eta : F \Rightarrow G$ is a natural isomorphism when η_X is an isomorphism for all $X \in \text{Obj}(\mathcal{C})$. By Definition 1.2.1, this means that, for each object X of $\mathcal{C}$, there exists a morphism $\eta'_X : G(X) \to F(X)$ such that

$$\eta'_X \circ \eta_X = \text{id}_{F(X)} \quad \text{and} \quad \eta_X \circ \eta'_X = \text{id}_{G(C)}.$$

To finish this proof, we will show that the family $\{\eta'_X \mid X \in \text{Obj}(\mathcal{C})\}$ is a natural transformation. To do that, let X and Y be objects of $\mathcal{C}$, let f be a morphism in $\text{Hom}_{\mathcal{C}}(X, Y)$, and notice that:

$$\begin{aligned}F(f) \circ \eta'_X &= \text{id}_{F(Y)} \circ F(f) \circ \eta'_X \\ &= \eta'_Y \circ \eta_Y \circ F(f) \circ \eta'_X \\ &= \eta'_Y \circ G(f) \circ \eta_X \circ \eta'_X \\ &= \eta'_Y \circ G(f) \circ \text{id}_{G(X)} \\ &= \eta'_Y \circ G(f).\end{aligned}$$

This shows that $\eta' : G \Rightarrow F$ is a natural transformation. The fact that $\eta' \circ \eta = \text{id}_F$ and $\eta \circ \eta' = \text{id}_G$ follows from the construction of η'. □

3.2. Equivalences of Categories

Equivalences of categories is the formal notion that captures the idea of categories that "behave in the same way". In this section, we will formally define equivalences of categories, provide some examples illustrating when two categories are equivalent or non-equivalent, and prove two results regarding equivalences of categories that will be used in the subsequent sections.

We begin with the abstract definition of an equivalence of categories.

Definition 3.2.1. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an *equivalence of categories* when there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms, η and ϵ, such that:

$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F \quad \text{and} \quad \epsilon : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}\,.$$

In this case, one says that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent categories* and denotes this relation by $\mathcal{C} \simeq \mathcal{D}$.

Notice that "equivalence" is a weaker notion than that of "isomorphism" of categories. Namely, an *isomorphism of categories* consists of a pair of functors, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, such that $G \circ F = \mathrm{Id}_{\mathcal{C}}$ and $F \circ G = \mathrm{Id}_{\mathcal{D}}$ (strict equality). This implies that, while isomorphisms of categories require a perfect one-to-one correspondence between objects and morphisms, equivalences allow for more flexibility by ignoring "inessential" differences, such as multiple isomorphic copies of objects. This difference makes equivalences of categories often more useful in practice.

We turn now to some examples that illustrate these concepts. We begin with the simplest example of an equivalence of categories, which incidentally this is also an isomorphism of categories, that of the identity functor.

Example 3.2.2. Every category $\mathcal{C}$ is equivalent to itself via the identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (constructed in Example 2.1.2). In fact, if we take $F = G = \mathrm{Id}_{\mathcal{C}}$, we have that $F \circ G = \mathrm{Id}_{\mathcal{C}} \circ \mathrm{Id}_{\mathcal{C}} = \mathrm{Id}_{\mathcal{C}}$ and $G \circ F = \mathrm{Id}_{\mathcal{C}} \circ \mathrm{Id}_{\mathcal{C}} = \mathrm{Id}_{\mathcal{C}}$. Hence, in this case, the natural isomorphisms η and ϵ can be chosen to be the identity natural transformations (constructed in Example 3.1.2).

While the identity functor is an equivalence of a category with itself, more interesting examples arise when comparing different categories.

Example 3.2.3. Let $\mathcal{C}$ be the category with one object, $\mathrm{Obj}(\mathcal{C}) = \{X\}$, and one morphism, $\mathrm{Mor}(\mathcal{C}) = \{\mathrm{id}_X\}$. Then, let $\mathcal{D}$ be the category with two objects, $\mathrm{Obj}(\mathcal{D}) = \{Y, Z\}$, and two non-identity morphisms, $\mathrm{Mor}(\mathcal{D}) = \{\mathrm{id}_Y, \mathrm{id}_Z, f, g\}$, where $f \in \mathrm{Hom}_{\mathcal{D}}(Y, Z)$, $g \in \mathrm{Hom}_{\mathcal{D}}(Z, Y)$, $f \circ g = \mathrm{id}_Z$ and $g \circ f = \mathrm{id}_Y$. These categories are equivalent and non-isomorphic.

To see this, define the functor $F : \mathcal{C} \to \mathcal{D}$ by choosing

$$F(X) = Y \quad \text{and} \quad F(\mathrm{id}_X) = \mathrm{id}_Y,$$

and define the functor $G : \mathcal{D} \to \mathcal{C}$ by choosing

$$G(Y) = G(Z) = X \quad \text{and} \quad G(\mathrm{id}_Y) = G(\mathrm{id}_Z) = G(f) = G(g) = \mathrm{id}_X\,.$$

To show that F and G are equivalences of categories, we must construct natural isomorphisms η and ϵ such that $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow (G \circ F)$ and $\epsilon : (F \circ G) \Rightarrow \mathrm{Id}_{\mathcal{D}}$.

Since $G \circ F$ is the identity functor on $\mathcal{C}$, we can choose η to be the identity natural transformation. Then, to construct ϵ, notice that the functor $F \circ G$ is given by:

$$(F \circ G)(Y) = (F \circ G)(Z) = Y,$$
$$(F \circ G)(\mathrm{id}_Y) = (F \circ G)(\mathrm{id}_Z) = (F \circ G)(f) = (F \circ G)(g) = \mathrm{id}_Y .$$

Hence, we must choose $\epsilon_Y \in \mathrm{Hom}_{\mathcal{D}}(Y, Y)$ and $\epsilon_Z \in \mathrm{Hom}_{\mathcal{D}}(Y, Z)$ satisfying the naturality conditions. Since $\mathrm{Hom}_{\mathcal{D}}(Y, Y) = \{\mathrm{id}_Y\}$ and $\mathrm{Hom}_{\mathcal{D}}(Y, Z) = \{f\}$, we must choose $\epsilon_Y = \mathrm{id}_Y$ and $\epsilon_Z = f$. To verify that this $\epsilon : (F \circ G) \Rightarrow \mathrm{Id}_{\mathcal{D}}$ is indeed a natural transformation, we write down the naturality conditions explicitly:

$$\epsilon_Y \circ (F \circ G)(\mathrm{id}_Y) = \mathrm{id}_Y \circ \mathrm{id}_Y = \mathrm{Id}_{\mathcal{D}}(\mathrm{id}_Y) \circ \epsilon_Y,$$
$$\epsilon_Z \circ (F \circ G)(\mathrm{id}_Z) = f \circ \mathrm{id}_Y = \mathrm{id}_Y \circ f = \mathrm{Id}_{\mathcal{D}}(\mathrm{id}_Z) \circ \epsilon_Z,$$
$$\epsilon_Z \circ (F \circ G)(f) = f \circ \mathrm{id}_Y = \mathrm{Id}_{\mathcal{D}}(f) \circ \epsilon_Y,$$
$$\epsilon_Y \circ (F \circ G)(g) = \mathrm{id}_Y \circ \mathrm{id}_Y = \mathrm{id}_Y = g \circ f = \mathrm{Id}_{\mathcal{D}}(g) \circ \epsilon_Z.$$

Moreover, notice that $\epsilon_Y = \mathrm{id}_Y$ and $\epsilon_Z = f$ are isomorphisms. This means that ϵ is in fact a natural isomorphism.

The argument above shows that $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories (and that F and G are equivalences). However, these categories are not isomorphic. In fact, since no functor $G : \mathcal{D} \to \mathcal{C}$ can be injective on objects and morphisms, there exists no functor $F : \mathcal{C} \to \mathcal{D}$ such that $(F \circ G)$ is injective on objects and morphisms. This implies that there exists no such functors F and G such that $(F \circ G)$ is equal to the identity functor $\mathrm{Id}_{\mathcal{D}}$.

Not all small categories are equivalent, however. The next example shows a case of explicit categories that are not equivalent.

Example 3.2.4. Let $\mathcal{C}$ be the category with one object and one morphism,

$$\mathrm{Obj}(\mathcal{C}) = \{X\} \quad \text{and} \quad \mathrm{Mor}(\mathcal{C}) = \{\mathrm{id}_X\}.$$

Then, let $\mathcal{D}$ be the category with two objects and only identity morphisms,

$$\mathrm{Obj}(\mathcal{D}) = \{Y, Z\} \quad \text{and} \quad \mathrm{Mor}(\mathcal{D}) = \{\mathrm{id}_Y, \mathrm{id}_Z\}.$$

These categories are not equivalent.

To see this, we explicitly construct all possible functors between these categories. We begin by constructing the unique functor from $\mathcal{D}$ to $\mathcal{C}$. In fact, notice that, since $\mathcal{C}$ has only one object, the only possible functor $G : \mathcal{D} \to \mathcal{C}$ is given by

$$G(Y) = G(Z) = X \quad \text{and} \quad G(\mathrm{id}_Y) = G(\mathrm{id}_Z) = \mathrm{id}_X\,.$$

Then, notice that there are exactly two functors from $\mathcal{C}$ to $\mathcal{D}$, determined by which object of $\mathcal{D}$ is assigned to the object X. In fact, one can define a functor $F_1 : \mathcal{C} \to \mathcal{D}$ by choosing

$$F_1(X) = Y \quad \text{and} \quad F_1(\mathrm{id}_X) = \mathrm{id}_Y,$$

and define a functor $F_2 : \mathcal{C} \to \mathcal{D}$ by choosing

$$F_2(X) = Z \quad \text{and} \quad F_2(\mathrm{id}_X) = \mathrm{id}_Z\,.$$

Now, recall from Definition 3.2.1 that, is order for $\mathcal{C}$ to be equivalent to $\mathcal{D}$, there must exist a natural transformation $\epsilon : F_1 \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$ or a natural transformation $\theta : F_2 \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$. In the first case, ϵ_Z should be a morphism in $\mathrm{Hom}_{\mathcal{D}}(Y, Z)$, which is empty; and in the second case, θ_Y should be a morphism in $\mathrm{Hom}_{\mathcal{D}}(Z, Y)$, which is also empty. This means that no such natural transformations exist, and thus that $\mathcal{C}$ is not equivalent to $\mathcal{D}$.

We close this section by showing that equivalence of categories is not just a property but an equivalence relation on categories. This is formalized in the following result.

Proposition 3.2.5. Equivalence of categories is an equivalence relation.

Proof. Begin by recalling from Example 3.2.2 that every category is equivalent to itself via the identity functor. This means that the relation $\simeq$ is reflexive.

Next, we will show that, if $\mathcal{C} \simeq \mathcal{D}$, then $\mathcal{D} \simeq \mathcal{C}$. To do that, we begin by recalling from Definition 3.2.1 that, if $\mathcal{C} \simeq \mathcal{D}$, then there exist: a functor $F : \mathcal{C} \to \mathcal{D}$, a functor $G : \mathcal{D} \to \mathcal{C}$, a natural isomorphism $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$, and a natural isomorphism $\epsilon : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$. By Proposition 3.1.4(c), this implies that there exist a natural isomorphism $\eta' : G \circ F \Rightarrow \mathrm{Id}_{\mathcal{C}}$ and a natural isomorphism $\epsilon' : \mathrm{Id}_{\mathcal{D}} \Rightarrow F \circ G$. This means exactly that $\mathcal{D} \simeq \mathcal{C}$, and shows that the relation $\simeq$ is symmetric.

To complete this proof, we will to show that, if $\mathcal{A} \simeq \mathcal{B}$ and $\mathcal{B} \simeq \mathcal{C}$, then $\mathcal{A} \simeq \mathcal{C}$. To do that, begin by assuming that $\mathcal{A} \simeq \mathcal{B}$ and $\mathcal{B} \simeq \mathcal{C}$. Then, recall that the equivalence $\mathcal{A} \simeq \mathcal{B}$ means that there exist: functors $F_1 : \mathcal{A} \to \mathcal{B}$ and $G_1 : \mathcal{B} \to \mathcal{A}$, and natural isomorphisms $\eta_1 : \mathrm{Id}_{\mathcal{A}} \Rightarrow (G_1 \circ F_1)$ and $\epsilon_1 : (F_1 \circ G_1) \Rightarrow \mathrm{Id}_{\mathcal{B}}$. Similarly, recall that the equivalence $\mathcal{B} \simeq \mathcal{C}$ means that there exist: functors $F_2 : \mathcal{B} \to \mathcal{C}$ and $G_2 : \mathcal{C} \to \mathcal{B}$, and natural isomorphisms $\eta_2 : \mathrm{Id}_{\mathcal{B}} \Rightarrow (G_2 \circ F_2)$ and $\epsilon_2 : (F_2 \circ G_2) \Rightarrow \mathrm{Id}_{\mathcal{C}}$. Now, to prove that $\mathcal{A} \simeq \mathcal{C}$, begin by noticing that $(F_2 \circ F_1) : \mathcal{A} \to \mathcal{C}$ and $(G_1 \circ G_2) : \mathcal{C} \to \mathcal{A}$ are functors (see Proposition 2.1.6). Thus, we only need to construct natural isomorphisms $\phi : \mathrm{Id}_{\mathcal{A}} \Rightarrow (G_1 \circ G_2 \circ F_2 \circ F_1)$ and $\psi : (F_2 \circ F_1 \circ G_1 \circ G_2) \Rightarrow \mathrm{Id}_{\mathcal{C}}$.

We will construct the families $\{\phi_A \mid A \in \mathrm{Obj}(\mathcal{A})\}$ and $\{\psi_C \mid C \in \mathrm{Obj}(\mathcal{C})\}$, and then show that they define the desired natural isomorphisms. Begin by defining, for each object A of $\mathcal{A}$ and for each object C of $\mathcal{C}$, the morphisms

$$\phi_A := G_1\left((\eta_2)_{F_1(A)}\right) \circ (\eta_1)_A \quad \text{and} \quad \psi_C := F_2\left((\epsilon_1)_{G_2(C)}\right) \circ (\epsilon_2)_C.$$

Now, we will show that the family $\{\phi_A \mid A \in \mathrm{Obj}(\mathcal{A})\}$ defines a natural isomorphism. The proof that $\{\psi_C \mid C \in \mathrm{Obj}(\mathcal{C})\}$ also defines a natural isomorphism is very similar. To unpack the definition of ϕ_A, begin by recalling that $(\eta_1)_A$ is an isomorphism, $(\eta_1)_A : A \to G_1(F_1(A))$. Then, recall that $(\eta_2)_B$ is also an isomorphism, $(\eta_2)_B : B \to G_2(F_2(B))$, for every object B of $\mathcal{B}$; in particular, for $B = F_1(A)$. Hence, $G((\eta_2)_{F_1(A)})$ is an isomorphism,

$$G_1\left((\eta_2)_{F_1(A)}\right) : G_1(F_1(A)) \to G_1(G_2(F_2(F_1(A)))),$$

and thus, ϕ_A is an isomorphism $\phi_A : A \to G_1(G_2(F_2(F_1(A))))$. To conclude that ϕ is a natural isomorphism, we only need to show that $\{\phi_A \mid A \in \mathrm{Obj}(\mathcal{A})\}$ is in fact a natural transformation. To that end, one can use the fact that G_1 is a functor and that η_2 is a natural transformation to verify that

$$\begin{aligned}
G_1(G_2(F_2(F_1(f)))) \circ \phi_A &= G_1(G_2(F_2(F_1(f)))) \circ G_1\left((\eta_2)_{F_1(A)}\right) \circ (\eta_1)_A \\
&= G_1\left(G_2(F_2(F_1(f))) \circ (\eta_2)_{F_1(A)}\right) \circ (\eta_1)_A \\
&= G_1\left((\eta_2)_{F_1(A')} \circ F_1(f)\right) \circ (\eta_1)_A \\
&= G_1\left((\eta_2)_{F_1(A')}\right) \circ G_1(F_1(f)) \circ (\eta_1)_A \\
&= G_1\left((\eta_2)_{F_1(A')}\right) \circ (\eta_1)_{A'} \circ f \\
&= \phi_{A'} \circ f,
\end{aligned}$$

for every pair of objects A, A' of $\mathcal{A}$ and every morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, A')$. This proves that $\mathcal{A} \simeq \mathcal{C}$, that is, that the relation $\simeq$ is also transitive, and finishes the proof that $\simeq$ is an equivalence relation. □

3.3. Products of Categories

Before proceeding, we review products in general, and in particular, products of categories. Recall from Definition 1.2.9 that if $\mathcal{C}$ is a category and A, B are two objects of $\mathcal{C}$, then the product of A and B in $\mathcal{C}$ is a triple (P, p_A, p_B), where P is an object of $\mathcal{C}$ and $p_A : P \to A$, $p_B : P \to B$ are morphisms satisfying a universal property.

For example, recall from Example 1.2.11 that the Cartesian product of sets, together with their canonical projections, serves as the product in the category of sets. Similarly, in the category of vector spaces over a fixed field, the direct sum and their corresponding projections serve as the product.

In this section, we construct the category of small categories, describe the product in this category, and prove that this product induces a functor satisfying certain properties. This construction will be used in the next section to define monoidal categories.

We begin by considering the category of small categories.

Example 3.3.1. Let **Cats** be the category whose objects are small categories, whose morphisms are functors between them, and whose composition is given by Proposition 2.1.6(a). This is indeed a category: for every (small) category, there exists an identity functor (see Example 2.1.2), and the composition of functors is associative (see Proposition 2.1.6(b)).

Next, we describe the product of two categories within the category of small categories constructed in the previous example.

Example 3.3.2. Consider the category of small categories, **Cats**, constructed in Example 3.3.1. We construct the product of two small categories $\mathcal{C}$ and $\mathcal{D}$ by providing a triple $(\mathcal{C} \times \mathcal{D}, p_{\mathcal{C}}, p_{\mathcal{D}})$, where $\mathcal{C} \times \mathcal{D}$ is an object (a category), and $p_{\mathcal{C}} : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $p_{\mathcal{D}} : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ are morphisms (functors) satisfying condition (iv) in Definition 1.2.9.

We begin by constructing $\mathcal{C} \times \mathcal{D}$. Let the objects of $\mathcal{C} \times \mathcal{D}$ be pairs (c, d), where c is an object of $\mathcal{C}$ and d is an object of $\mathcal{D}$; that is, $\mathrm{Obj}(\mathcal{C} \times \mathcal{D})$ is the Cartesian product $\mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$. Given two objects (c, d) and (c', d') of $\mathcal{C} \times \mathcal{D}$, a morphism between them is a pair (f, g), where $f : c \to c'$ is a morphism in $\mathcal{C}$ and $g : d \to d'$ is a morphism in $\mathcal{D}$; that is,

$$\mathrm{Hom}_{\mathcal{C} \times \mathcal{D}}((c, d), (c', d')) = \mathrm{Hom}_{\mathcal{C}}(c, c') \times \mathrm{Hom}_{\mathcal{D}}(d, d').$$

Composition of morphisms in $\mathcal{C} \times \mathcal{D}$ is defined component-wise; that is,

$$(f', g') \circ_{\mathcal{C}\times\mathcal{D}} (f, g) := (f' \circ_{\mathcal{C}} f,\ g' \circ_{\mathcal{D}} g)\,,$$

for all $(f, g) \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((c, d), (c', d'))$ and $(f', g') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((c', d'), (c'', d''))$.

We now verify that this construction of $\mathcal{C} \times \mathcal{D}$ is indeed a small category, i.e., an object of **Cats**. First, notice that the objects of $\mathcal{C} \times \mathcal{D}$ forms a set because both $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Obj}(\mathcal{D})$ are sets. Next, define the identity morphisms in $\mathcal{C} \times \mathcal{D}$ by

$$\mathrm{id}_{(c,d)} = (\mathrm{id}_c, \mathrm{id}_d) \qquad \text{for all } (c, d) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D}).$$

Indeed,

$$(f, g) \circ (\mathrm{id}_c, \mathrm{id}_d) = (f \circ \mathrm{id}_c,\ g \circ \mathrm{id}_d) = (f, g),$$

and

$$(\mathrm{id}_{c'}, \mathrm{id}_{d'}) \circ (f, g) = (\mathrm{id}_{c'} \circ f,\ \mathrm{id}_{d'} \circ g) = (f, g),$$

for every morphism $(f, g) \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((c, d), (c', d'))$. Associativity of the composition of $\mathcal{C} \times \mathcal{D}$ follows directly from its component-wise definition and the associativity of compositions of $\mathcal{C}$ and $\mathcal{D}$.

Next, we construct the functors $p_{\mathcal{C}} : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $p_{\mathcal{D}} : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$. Define $p_{\mathcal{C}}$ by assigning:

- to each object (c, d) of $\mathcal{C} \times \mathcal{D}$, the object c of $\mathcal{C}$;
- to each morphism $(f, g) \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((c, d), (c', d'))$, the morphism f in $\mathcal{C}$.

To verify that $p_{\mathcal{C}}$ is indeed a functor, we will check that it satisfies conditions (i) and (ii) of Definition 2.1.1:

(i) For any object (c, d) of $\mathcal{C} \times \mathcal{D}$,

$$p_{\mathcal{C}}(\mathrm{id}_{(c,d)}) = p_{\mathcal{C}}(\mathrm{id}_c, \mathrm{id}_d) = \mathrm{id}_c\,.$$

(ii) For any composable morphisms (f, g) and (f', g') as above,

$$p_{\mathcal{C}}\left((f', g') \circ (f, g)\right) = p_{\mathcal{C}}(f' \circ f,\ g' \circ g) = f' \circ f = p_{\mathcal{C}}(f', g') \circ p_{\mathcal{C}}(f, g).$$

This concludes the verification that $p_{\mathcal{C}}$ is a functor. Similarly, define $p_{\mathcal{D}}$ by assigning:

- to each object (c, d) of $\mathcal{C} \times \mathcal{D}$, the object d of $\mathcal{D}$;
- to each morphism $(f, g) \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((c, d), (c', d'))$, the morphism g in $\mathcal{D}$.

The verification that $p_{\mathcal{D}}$ is a functor is very similar to the one for $p_{\mathcal{C}}$.

To complete this example, we verify that the triple $(\mathcal{C} \times \mathcal{D}, p_{\mathcal{C}}, p_{\mathcal{D}})$ satisfies the universal property of the product in **Cats** (see Definition 1.2.9). Namely, we will verify that, if $\mathcal{X}$ is a small category for which there exist functors $F_{\mathcal{C}} : \mathcal{X} \to \mathcal{C}$ and $F_{\mathcal{D}} : \mathcal{X} \to \mathcal{D}$, then there exists a unique functor $F : \mathcal{X} \to \mathcal{C} \times \mathcal{D}$ such that

$$p_{\mathcal{C}} \circ F = F_{\mathcal{C}} \quad \text{and} \quad p_{\mathcal{D}} \circ F = F_{\mathcal{D}}.$$

Define F by assigning:

- to each object x of $\mathcal{X}$, the object $(F_{\mathcal{C}}(x), F_{\mathcal{D}}(x))$ of $\mathcal{C} \times \mathcal{D}$;
- to each morphism $f \in \mathrm{Hom}_{\mathcal{X}}(x, y)$, the morphism $(F_{\mathcal{C}}(f), F_{\mathcal{D}}(f))$.

The fact that $p_{\mathcal{C}} \circ F = F_{\mathcal{C}}$ and $p_{\mathcal{D}} \circ F = F_{\mathcal{D}}$ follows directly from the definition of F. Thus, we are left to verify that F is indeed a functor. To do that, we will check that it satisfies conditions (i) and (ii) of Definition 2.1.1. Indeed:

(i) For each object x of $\mathcal{X}$, using the fact that $F_{\mathcal{C}}$ and $F_{\mathcal{D}}$ are functors and the form of the identity morphism of $\mathcal{C} \times \mathcal{D}$ given above, we have:

$$F(\mathrm{id}_x) = (F_{\mathcal{C}}(\mathrm{id}_x), F_{\mathcal{D}}(\mathrm{id}_x)) = (\mathrm{id}_{F_{\mathcal{C}}(x)}, \mathrm{id}_{F_{\mathcal{D}}(x)}) = \mathrm{id}_{(F_{\mathcal{C}}(x), F_{\mathcal{D}}(x))} = \mathrm{id}_{F(x)} .$$

(ii) To verify the second condition, let x, y, z be objects of $\mathcal{X}$, $f \in \mathrm{Hom}_{\mathcal{X}}(x, y)$ and $g \in \mathrm{Hom}_{\mathcal{X}}(y, z)$. Using the definition of F, the fact that $F_{\mathcal{C}}$ and $F_{\mathcal{D}}$ are functors, and the form of the composition on $\mathcal{C} \times \mathcal{D}$, we have:

$$\begin{aligned} F(g \circ f) &= (F_{\mathcal{C}}(g \circ f),\ F_{\mathcal{D}}(g \circ f)) \\ &= (F_{\mathcal{C}}(g) \circ F_{\mathcal{C}}(f),\ F_{\mathcal{D}}(g) \circ F_{\mathcal{D}}(f)) \\ &= (F_{\mathcal{C}}(g),\ F_{\mathcal{D}}(g)) \circ (F_{\mathcal{C}}(f),\ F_{\mathcal{D}}(f)) \\ &= F(g) \circ F(f). \end{aligned}$$

This concludes the proof that the triple $(\mathcal{C} \times \mathcal{D}, p_{\mathcal{C}}, p_{\mathcal{D}})$ satisfies the defining conditions of the product in **Cats**. Thus, we conclude that $(\mathcal{C} \times \mathcal{D}, p_{\mathcal{C}}, p_{\mathcal{D}})$ is the product of the categories $\mathcal{C}$ and $\mathcal{D}$.

We close this section by proving that products define a functor satisfying certain properties. These properties will become axioms of monoidal categories in the next section.

Proposition 3.3.3. Let $\mathcal{C}$ be a category.

(a) If $\mathcal{C}$ has finite products, then there exists a functor $\Pi : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ induced by these products.

(b) There exists a natural isomorphism $\alpha : \Pi \circ (\Pi \times \mathrm{id}_{\mathcal{C}}) \Rightarrow \Pi \circ (\mathrm{id}_{\mathcal{C}} \times \Pi)$.

(c) If $\mathcal{C}$ has a terminal object $\mathbb{1}$, then there exist natural isomorphisms

$$\{\lambda_c : \mathbb{1} \times c \to c \mid c \in \mathrm{Obj}(\mathcal{C})\} \quad \text{and} \quad \{\varrho_c : c \times \mathbb{1} \to c \mid c \in \mathrm{Obj}(\mathcal{C})\}.$$

(d) For each quadruple of objects a, b, c, d of $\mathcal{C}$, we have

$$\alpha_{a,b,c\times d} \circ \alpha_{a\times b,c,d} = (\mathrm{id}_a \times \alpha_{b,c,d}) \circ \alpha_{a,b\times c,d} \circ (\alpha_{a,b,c} \times \mathrm{id}_d).$$

(e) For each pair of objects a, b of $\mathcal{C}$, we have

$$(\mathrm{id}_a \times \lambda_b) \circ \alpha_{a,\mathbb{1},b} = (\varrho_a \times \mathrm{id}_b).$$

Proof. (a) We begin by defining Π. Since the category $\mathcal{C}$ is assumed to have finite products, for any object (a, b) of $\mathcal{C} \times \mathcal{C}$, the product of a and b in $\mathcal{C}$ is a triple $(a \times b, p_a, p_b)$ (see Definition 1.2.9). We will assign $\Pi(a, b) := a \times b$.

Now, consider a morphism $(f : a \to a',\ g : b \to b')$ of $\mathcal{C} \times \mathcal{C}$. The product $\Pi(f, g)$ will be defined as the unique morphism $f \times g : a \times b \to a' \times b'$ such that $p_{a'} \circ (f \times g) = (f \circ p_a)$ and $p_{b'} \circ (f \times g) = (g \circ p_b)$. The well-definiteness (existence and uniqueness) of $f \times g$ follows from the universal property of products (see Definition 1.2.9).

Now, we will verify that Π, as defined above, is a functor. That is, we will verify that Π satisfies conditions (i) and (ii) of Definition 2.1.1:

(i) Given an object (a, b) of $\mathcal{C} \times \mathcal{C}$, we must check that $\Pi(\mathrm{id}_{(a,b)}) = \mathrm{id}_{\Pi(a,b)}$. Since $\mathrm{id}_{\Pi(a,b)} = \mathrm{id}_{a\times b}$ and $\mathrm{id}_{(a,b)} = (\mathrm{id}_a, \mathrm{id}_b)$, this is equivalent to the equations:

$$p_a \circ \mathrm{id}_{a\times b} = p_a = \mathrm{id}_a \circ p_a \quad \text{and} \quad p_b \circ \mathrm{id}_{a\times b} = p_b = \mathrm{id}_b \circ p_b.$$

(ii) Given two morphisms of $\mathcal{C} \times \mathcal{C}$, namely $(f, g) : (a, b) \to (a', b')$ and $(f', g') : (a', b') \to (a'', b'')$, we must check that

$$\Pi((f', g') \circ (f, g)) = \Pi(f', g') \circ \Pi(f, g).$$

Since $(f', g') \circ (f, g) = (f' \circ f,\ g' \circ g)$, this is equivalent to the equations:

$$\begin{aligned} p_{a''} \circ ((f' \times g') \circ (f \times g)) &= (p_{a''} \circ (f' \times g')) \circ (f \times g) \\ &= (f' \circ p_{a'}) \circ (f \times g) \\ &= f' \circ (p_{a'} \circ (f \times g)) \\ &= f' \circ (f \circ p_a) \\ &= (f' \circ f) \circ p_a \end{aligned}$$

and

$$\begin{aligned} p_{b''} \circ ((f' \times g') \circ (f \times g)) &= (p_{b''} \circ (f' \times g')) \circ (f \times g) \\ &= (g' \circ p_{b'}) \circ (f \times g) \\ &= g' \circ (p_{b'} \circ (f \times g)) \\ &= g' \circ (g \circ p_b) \\ &= (g' \circ g) \circ p_b. \end{aligned}$$

This completes the proof that Π is a functor.

(b) We want to construct a natural isomorphism

$$\alpha : \Pi \circ (\Pi \times \mathrm{id}_{\mathcal{C}}) \Rightarrow \Pi \circ (\mathrm{id}_{\mathcal{C}} \times \Pi).$$

Since

$$(\Pi \circ (\Pi \times \mathrm{id}))(a, b, c) = \Pi((a \times b), c) = (a \times b) \times c$$

and

$$(\Pi \circ (\mathrm{id} \times \Pi))(a, b, c) = \Pi(a, (b \times c)) = a \times (b \times c),$$

for every triple of objects a, b, c of $\mathcal{C}$, this is equivalent to constructing a family of isomorphisms

$$\{\alpha_{a,b,c} : (a \times b) \times c \to a \times (b \times c) \mid a, b, c \in \mathrm{Obj}(\mathcal{C})\}$$

that satisfies naturality conditions.

Fix a triple of objects $a, b, c \in \mathrm{Obj}(\mathcal{C})$. Using the universal property of the product $a \times (b \times c)$, the morphism $\alpha_{a,b,c}$ is uniquely determined by a pair of morphisms:

$$f_a : (a \times b) \times c \to a \quad \text{and} \quad f_{b,c} : (a \times b) \times c \to b \times c.$$

Similarly, using the universal property of the product $b \times c$, the morphism $f_{b,c}$ is uniquely determined by a pair of morphisms

$$f_b : (a \times b) \times c \to b \quad \text{and} \quad f_c : (a \times b) \times c \to c.$$

Choose $f_a = (p_a \circ p_{a\times b})$, $f_b = (p_b \circ p_{a\times b})$ and $f_c = p_c$. Thus, we will define $\alpha_{a,b,c}$ to be the unique morphism in $\mathrm{Hom}_{\mathcal{C}}((a \times b) \times c,\, a \times (b \times c))$ such that

$$p_a \circ \alpha_{a,b,c} = f_a = p_a \circ p_{a\times b} \quad \text{and} \quad p_{b\times c} \circ \alpha_{a,b,c} = f_{b,c},$$

where $f_{b,c}$ is the unique morphism in $\mathrm{Hom}_{\mathcal{C}}((a \times b) \times c,\, b \times c)$ such that

$$p_b \circ f_{b,c} = f_b = p_b \circ p_{a\times b} \quad \text{and} \quad p_c \circ f_{b,c} = f_c = p_c.$$

To show that $\alpha_{a,b,c}$ is an isomorphism, we will construct its inverse. Let $\tilde{\alpha}_{a,b,c}$ be the unique morphism in $\mathrm{Hom}_{\mathcal{C}}(a \times (b \times c),\, (a \times b) \times c)$ such that:

$$(p_a \circ p_{a\times b}) \circ \alpha_{a,b,c} = p_a, \qquad (p_b \circ p_{a\times b}) \circ \alpha_{a,b,c} = p_b \circ p_{b\times c},$$
$$p_c \circ \tilde{\alpha}_{a,b,c} = p_c \circ p_{b\times c}.$$

By construction, $\alpha_{a,b,c} \circ \tilde{\alpha}_{a,b,c}$ is a morphism in $\mathrm{Hom}_{\mathcal{C}}(a \times (b \times c),\, a \times (b \times c))$ such that

$$p_a \circ (\alpha_{a,b,c} \circ \tilde{\alpha}_{a,b,c}) = (p_a \circ \alpha_{a,b,c}) \circ \tilde{\alpha}_{a,b,c} = (p_a \circ p_{a\times b}) \circ \tilde{\alpha}_{a,b,c} = p_a,$$
$$(p_b \circ p_{b\times c}) \circ (\alpha_{a,b,c} \circ \tilde{\alpha}_{a,b,c}) = (p_b \circ p_{a\times b}) \circ \tilde{\alpha}_{a,b,c} = p_b \circ p_{b\times c},$$
$$(p_c \circ p_{b\times c}) \circ (\alpha_{a,b,c} \circ \tilde{\alpha}_{a,b,c}) = p_c \circ \tilde{\alpha}_{a,b,c} = p_c \circ p_{b\times c}.$$

Since

$$p_a \circ \mathrm{id}_{a\times(b\times c)} = p_a, \qquad (p_b \circ p_{b\times c}) \circ \mathrm{id}_{a\times(b\times c)} = p_b \circ p_{b\times c},$$
$$(p_c \circ p_{b\times c}) \circ \mathrm{id}_{a\times(b\times c)} = p_c \circ p_{b\times c},$$

the universal property of $a \times (b \times c)$ implies that $\alpha_{a,b,c} \circ \tilde{\alpha}_{a,b,c} = \mathrm{id}_{a\times(b\times c)}$. Similarly, we can verify that $\tilde{\alpha}_{a,b,c} \circ \alpha_{a,b,c} = \mathrm{id}_{(a\times b)\times c}$. This implies that $\alpha_{a,b,c}$ is an isomorphism.

Finally, to show that α satisfies the naturality conditions, consider three morphisms, $f : a \to a'$, $g : b \to b'$ and $h : c \to c'$, in $\mathcal{C}$. We need to verify that

$$(f \times (g \times h)) \circ \alpha_{a,b,c} = \alpha_{a',b',c'} \circ ((f \times g) \times h).$$

Notice that

$$\begin{aligned}p_{a'} \circ ((f \times (g \times h)) \circ \alpha_{a,b,c}) &= (p_{a'} \circ (f \times (g \times h))) \circ \alpha_{a,b,c} \\ &= (f \circ p_a) \circ \alpha_{a,b,c} \\ &= f \circ (p_a \circ \alpha_{a,b,c}) \\ &= f \circ (p_a \circ p_{a \times b}) \\ &= (f \circ p_a) \circ p_{a \times b} \\ &= (p_{a'} \circ (f \times g)) \circ p_{a \times b} \\ &= p_{a'} \circ ((f \times g) \circ p_{a \times b}) \\ &= p_{a'} \circ (p_{a' \times b'} \circ ((f \times g) \times h)) \\ &= (p_{a'} \circ p_{a' \times b'}) \circ ((f \times g) \times h) \\ &= (p_{a'} \circ \alpha_{a',b',c'}) \circ ((f \times g) \times h) \\ &= p_{a'} \circ (\alpha_{a',b',c'} \circ ((f \times g) \times h)) \,.\end{aligned}$$

Similarly, we can verify that

$$\begin{aligned}&(p_{b'} \circ p_{b' \times c'}) \circ ((f \times (g \times h)) \circ \alpha_{a,b,c}) \\ &\qquad = (p_{b'} \circ p_{b' \times c'}) \circ (\alpha_{a',b',c'} \circ ((f \times g) \times h))\end{aligned}$$

and

$$\begin{aligned}&(p_{c'} \circ p_{b' \times c'}) \circ ((f \times (g \times h)) \circ \alpha_{a,b,c}) \\ &\qquad = (p_{c'} \circ p_{b' \times c'}) \circ (\alpha_{a',b',c'} \circ ((f \times g) \times h)) \,.\end{aligned}$$

The constructions of α and of the product of morphisms in $\mathcal{C}$ via universal properties, and these identities imply that

$$(f \times (g \times h)) \circ \alpha_{a,b,c} = \alpha_{a',b',c'} \circ ((f \times g) \times h).$$

This shows that α is a natural transformation and concludes the proof of this part.

(c) We want to construct natural isomorphisms λ and ϱ. To do that, let c be an object of $\mathcal{C}$, let $\mathbf{1}$ be a terminal object of $\mathcal{C}$, and recall from Definition 1.2.9 that the product of the objects c and $\mathbf{1}$ in $\mathcal{C}$ is a triple $(c \times \mathbf{1}, p_c, p_{\mathbf{1}})$, where $c \times \mathbf{1}$ is an object of $\mathcal{C}$ and $p_c : c \times \mathbf{1} \to c$, $p_{\mathbf{1}} : c \times \mathbf{1} \to \mathbf{1}$ are morphisms of $\mathcal{C}$. We will define $\lambda_c = p_c$ for all $c \in \mathrm{Obj}(\mathcal{C})$.

To show that λ_c is an isomorphism for every object c of $\mathcal{C}$, we will use the fact that $\mathbf{1}$ is a terminal object of $\mathcal{C}$. In fact, fix an object $c \in \mathrm{Obj}(\mathcal{C})$, and recall from Definition 1.2.4 that, for each object x of $\mathcal{C}$, there exists a unique morphism $t_x : x \to \mathbf{1}$; in particular, for c and for $c \times \mathbf{1}$. This means that there exist morphisms $t_c : c \to \mathbf{1}$ and $\mathrm{id}_c : c \to c$. Thus, the universal property that $(c \times \mathbf{1}, p_c, p_{\mathbf{1}})$ satisfies (see Definition 1.2.9) implies that, there exists a unique morphism $\mu_c : c \to c \times \mathbf{1}$ such that $p_c \circ \mu_c = \mathrm{id}_c$ and $p_{\mathbf{1}} \circ \mu_c = t_c$. This first equation implies that

$$\lambda_c \circ \mu_c = p_c \circ \mu_c = \mathrm{id}_c\,.$$

It also implies that $\mu_c \circ \lambda_c$ is a morphism $(c \times \mathbf{1}) \to (c \times \mathbf{1})$ such that

$$p_c \circ (\mu_c \circ \lambda_c) = p_c \circ (\mu_c \circ p_c) = (p_c \circ \mu_c) \circ p_c = \mathrm{id}_c \circ p_c = p_c$$

and $p_{\mathbf{1}} \circ (\mu_c \circ \lambda_c) = t_{c\times\mathbf{1}}$. Since $\mathrm{id}_{c\times\mathbf{1}}$ is also a morphism $(c \times \mathbf{1}) \to (c \times \mathbf{1})$ that satisfies

$$p_c \circ \mathrm{id}_{c\times\mathbf{1}} = p_c \quad \text{and} \quad p_{\mathbf{1}} \circ \mathrm{id}_{c\times\mathbf{1}} = t_{c\times\mathbf{1}},$$

the universal property of $(c \times \mathbf{1}, p_c, p_{\mathbf{1}})$ implies that $\mu_c \circ \lambda_c = \mathrm{id}_{c\times\mathbf{1}}$. This shows that $\lambda_c \circ \mu_c = \mathrm{id}_c$ and $\mu_c \circ \lambda_c = \mathrm{id}_{c\times\mathbf{1}}$, and as a consequence, that λ_c is an isomorphism.

Now, we will verify that $\lambda = \{\lambda_p \mid p \in \mathrm{Obj}(\mathcal{C})\}$ is a natural transformation, that is, that it the satisfies naturality conditions. To do that, let $f : c \to c'$ be a morphism of $\mathcal{C}$. From the definition of λ and the definition of the product of morphisms, we see that

$$\lambda_{c'} \circ (\mathrm{id}_{\mathbf{1}} \times f) = p_{c'} \circ (\mathrm{id}_{\mathbf{1}} \times f) = f \circ p_c = f \circ \lambda_c.$$

This shows that λ is a natural transformation, and completes the proof that λ is a natural isomorphism. The construction of ϱ and the proof that ϱ is a natural isomorphism are completely analogous.

(d) We want to prove that

$$\alpha_{a,b,c\times d} \circ \alpha_{a\times b,c,d} = (\mathrm{id}_a \times \alpha_{b,c,d}) \circ \alpha_{a,b\times c,d} \circ (\alpha_{a,b,c} \times \mathrm{id}_d).$$

Since they are morphisms in $\mathrm{Hom}_{\mathcal{C}}(((a\times b)\times c)\times d,\ a\times(b\times(c\times d)))$, we will use the universal properties of products and compare their projections. In fact, using the definition of α (see item (b)) and of products of morphisms (see item (a)), we obtain the following identities:

$$\begin{aligned}
&p_a \circ \alpha_{a,b,c\times d} \circ \alpha_{a\times b,c,d} \\
&\quad = p_a \circ p_{a\times b} \circ p_{(a\times b)\times c} \\
&\quad = p_a \circ (\mathrm{id}_a \times \alpha_{b,c,d}) \circ \alpha_{a,b\times c,d} \circ (\alpha_{a,b,c} \times \mathrm{id}_d), \\
&p_b \circ p_{b\times(c\times d)} \circ \alpha_{a,b,c\times d} \circ \alpha_{a\times b,c,d} \\
&\quad = p_b \circ p_{a\times b} \circ p_{(a\times b)\times c} \\
&\quad = p_b \circ p_{b\times(c\times d)} \circ (\mathrm{id}_a \times \alpha_{b,c,d}) \circ \alpha_{a,b\times c,d} \circ (\alpha_{a,b,c} \times \mathrm{id}_d), \\
&p_c \circ p_{c\times d} \circ p_{b\times(c\times d)} \circ \alpha_{a,b,c\times d} \circ \alpha_{a\times b,c,d} \\
&\quad = p_c \circ p_{(a\times b)\times c} \\
&\quad = p_c \circ p_{c\times d} \circ p_{b\times(c\times d)} \circ (\mathrm{id}_a \times \alpha_{b,c,d}) \circ \alpha_{a,b\times c,d} \circ (\alpha_{a,b,c} \times \mathrm{id}_d), \\
&p_d \circ p_{c\times d} \circ p_{b\times(c\times d)} \circ \alpha_{a,b,c\times d} \circ \alpha_{a\times b,c,d} \\
&\quad = p_d \\
&\quad = p_d \circ p_{c\times d} \circ p_{b\times(c\times d)} \circ (\mathrm{id}_a \times \alpha_{b,c,d}) \circ \alpha_{a,b\times c,d} \circ (\alpha_{a,b,c} \times \mathrm{id}_d),
\end{aligned}$$

Since these projections are equal, the universal property of $a\times(b\times(c\times d))$ implies that $\alpha_{a,b,c\times d} \circ \alpha_{a\times b,c,d}$ and $(\mathrm{id}_a \times \alpha_{b,c,d}) \circ \alpha_{a,b\times c,d} \circ (\alpha_{a,b,c} \times \mathrm{id}_d)$ are both equal to the unique morphism in $((a\times b)\times c)\times d \to a\times(b\times(c\times d))$ that satisfy these identities. This shows that

$$\alpha_{a,b,c\times d} \circ \alpha_{a\times b,c,d} = (\mathrm{id}_a \times \alpha_{b,c,d}) \circ \alpha_{a,b\times c,d} \circ (\alpha_{a,b,c} \times \mathrm{id}_d).$$

(e) Let a, b, c be a triple of objects of $\mathcal{C}$. We want to prove that

$$(\mathrm{id}_a \times \lambda_b) \circ \alpha_{a,b,c} = \varrho_a \times \mathrm{id}_b\,.$$

To do that, recall from the definition of the product of morphisms in $\mathcal{C}$ (see item (a)) that $\varrho_a \times \mathrm{id}_b$ is the unique morphism in $\mathrm{Hom}_{\mathcal{C}}((a\times\mathbf{1})\times b,\ a\times b)$ such that

$$p_a \circ (\varrho_a \times \mathrm{id}_b) = \varrho_a \circ p_{a\times\mathbf{1}} \quad \text{and} \quad p_b \circ (\varrho_a \times \mathrm{id}_b) = \mathrm{id}_b \circ p_b.$$

Thus, to show that $(\mathrm{id}_a \times \lambda_b) \circ \alpha_{a,b,c} = \varrho_a \times \mathrm{id}_b$ is equivalent to show that

$$p_a \circ ((\mathrm{id}_a \times \lambda_b) \circ \alpha_{a,b,c}) = \varrho_a \circ p_{a\times\mathbf{1}}$$
$$\text{and}$$
$$p_b \circ ((\mathrm{id}_a \times \lambda_b) \circ \alpha_{a,b,c}) = \mathrm{id}_b \circ p_b.$$

The first identity follows from the definition of the product of morphisms in $\mathcal{C}$, the definition of α (see item (b)) and the definition of ϱ (see item (c)):

$$\begin{aligned} p_a \circ ((\mathrm{id}_a \times \lambda_b) \circ \alpha_{a,\mathbf{1},b}) &= (p_a \circ (\mathrm{id}_a \times \lambda_b)) \circ \alpha_{a,\mathbf{1},b} \\ &= (\mathrm{id}_a \circ p_a) \circ \alpha_{a,\mathbf{1},b} \\ &= p_a \circ \alpha_{a,\mathbf{1},b} \\ &= p_a \circ p_{a\times\mathbf{1}} \\ &= \varrho_a \circ p_{a\times\mathbf{1}}. \end{aligned}$$

Similarly, the second identity follows from the definition of the product of morphisms in $\mathcal{C}$, the definition of α (see item (b)) and the definition of λ (see item (c)):

$$\begin{aligned} p_b \circ ((\mathrm{id}_a \times \lambda_b) \circ \alpha_{a,\mathbf{1},b}) &= (p_b \circ (\mathrm{id}_a \times \lambda_b)) \circ \alpha_{a,\mathbf{1},b} \\ &= (\lambda_b \circ p_{\mathbf{1}\times b}) \circ \alpha_{a,\mathbf{1},b} \\ &= (p_b \circ p_{\mathbf{1}\times b}) \circ \alpha_{a,\mathbf{1},b} \\ &= p_b. \end{aligned}$$

Since $(\mathrm{id}_a \times \lambda_b) \circ \alpha_{a,\mathbf{1},b}$ satisfies these identities, we conclude that it is equal to $\varrho \times \mathrm{id}_b$. ☐

3.4. Monoidal Categories

Monoidal categories are categories with a structure that mimics that of products (as seen in the previous section) and tensor products on vector spaces. In this section, we will define monoidal categories and provide several examples that illustrate their abstract definition.

Definition 3.4.1. A category $\mathcal{C}$ is said to be *monoidal* when it is equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying the following conditions:

(i) There is a natural isomorphism

$$\{\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z) \mid x, y, z \in \mathrm{Obj}(\mathcal{C})\}.$$

(ii) There is an object $\mathbf{1} \in \mathrm{Obj}(\mathcal{C})$ such that, there exist natural isomorphisms

$$\{\lambda_x : \mathbf{1} \otimes x \to x \mid x \in \mathrm{Obj}(\mathcal{C})\} \quad \text{and} \quad \{\varrho_x : x \otimes \mathbf{1} \to x \mid x \in \mathrm{Obj}(\mathcal{C})\}.$$

(iii) For every quadruple (a, b, c, d) of objects of $\mathcal{C}$, we have

$$(\mathrm{id}_a \otimes \alpha_{b,c,d}) \circ \alpha_{a,b\otimes c,d} \circ (\alpha_{a,b,c} \otimes \mathrm{id}_d) = \alpha_{a,b,c\otimes d} \circ \alpha_{a\otimes b,c,d}.$$

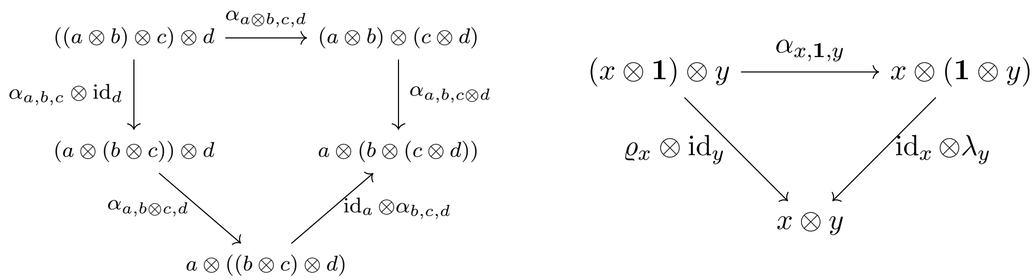


FIGURE 3.4.1. Diagrams of pentagon and triangle identities

(iv) For every pair (x, y) of objects of $\mathcal{C}$, we have

$$(\mathrm{id}_x \otimes \lambda_y) \circ \alpha_{x,\mathbf{1},y} = \varrho_x \otimes \mathrm{id}_y \,.$$

In this case, the functor $\otimes$ is called *tensor product*, the object $\mathbf{1}$ is called *identity object*, the equality in item (iii) is called *pentagon identity*, and the one in item (iv) is called *triangle identity* (see Figure 3.4.1).

To illustrate the abstract definition above, we will consider a few concrete examples of categories with a monoidal structure and a category with no monoidal structure. We begin with the smallest monoidal category possible.

Example 3.4.2. Consider the smallest category possible, that is, the category $\mathcal{C}$ with one object, $\mathrm{Obj}(\mathcal{C}) = \{\bullet\}$, one morphism $\mathrm{Mor}(\mathcal{C}) = \{\mathrm{id}_\bullet\}$, and trivial composition, $\mathrm{id}_\bullet \circ \mathrm{id}_\bullet = \mathrm{id}_\bullet$. In order to introduce a monoidal structure on this category, we must define a tensor functor.

To that end, begin by noticing that the category $\mathcal{C} \times \mathcal{C}$ also has one object, $\mathrm{Obj}(\mathcal{C}\times\mathcal{C}) = \{(\bullet, \bullet)\}$, and one morphism, $\mathrm{Mor}(\mathcal{C}\times\mathcal{C}) = \{\mathrm{id}_{(\bullet,\bullet)}\} = \{(\mathrm{id}_\bullet, \mathrm{id}_\bullet)\}$ (see Example 3.3.2). Hence, there exists only one functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, namely, the functor given by

$$\bullet \otimes \bullet = \bullet \quad \text{and} \quad \mathrm{id}_\bullet \otimes \mathrm{id}_\bullet = \mathrm{id}_\bullet \,.$$

In order to show that $(\mathcal{C}, \otimes)$ is a monoidal category, we will verify that the conditions (i)-(iv) are satisfied.

(i) Since $\bullet$ is the only object of $\mathcal{C}$, the only triple of objects of $\mathcal{C}$ is $(\bullet, \bullet, \bullet)$. Hence, in this case, the natural transformation α consists of a morphism, $\alpha_{\bullet,\bullet,\bullet}$. Moreover, since $\bullet \otimes \bullet = \bullet$, then

$$(\bullet \otimes \bullet) \otimes \bullet = \bullet \otimes \bullet = \bullet \quad \text{and} \quad \bullet \otimes (\bullet \otimes \bullet) = \bullet \otimes \bullet = \bullet.$$

Thus, in this case, we can choose $\alpha_{\bullet,\bullet,\bullet} = \mathrm{id}_\bullet$ to be the isomorphism

$$\alpha_{\bullet,\bullet,\bullet} : (\bullet \otimes \bullet) \otimes \bullet \to \bullet \otimes (\bullet \otimes \bullet).$$

(ii) Since $\mathrm{Obj}(\mathcal{C}) = \{\bullet\}$, then $\not\Vdash$ must be $\bullet$. Moreover, the natural transformation λ will consist of a unique morphism,

$$\lambda_\bullet : \mathbf{1} \otimes \bullet \to \bullet,$$

and similarly, the natural transformation ϱ will consist of a unique morphism,

$$\varrho_\bullet : \bullet \otimes \mathbf{1} \to \bullet.$$

Now, since $\mathbf{1} \otimes \bullet = \bullet \otimes \bullet = \bullet$, we can choose the morphism $\lambda_\bullet$ to be $\mathrm{id}_\bullet$, which is an isomorphism

$$\mathrm{id}_\bullet = \lambda_\bullet : \mathbf{1} \otimes \bullet \to \bullet.$$

Similarly, since $\bullet \otimes \mathbf{1} = \bullet \otimes \bullet = \bullet$, we can also choose the morphism $\varrho_\bullet$ to be $\mathrm{id}_\bullet$, which is an isomorphism

$$\mathrm{id}_\bullet = \varrho_\bullet : \bullet \otimes \mathbf{1} \to \bullet.$$

(iii) Since $\mathrm{Obj}(\mathcal{C}) = \{\bullet\}$, then the only quadruple of objects of $\mathcal{C}$ is $(\bullet, \bullet, \bullet, \bullet)$. Moreover, since $\bullet \otimes \bullet = \bullet$ and $\alpha_{\bullet,\bullet,\bullet} = \mathrm{id}$, then the left-hand side of the pentagon identity is

$$\begin{aligned}
(\mathrm{id}_\bullet \otimes \alpha_{\bullet,\bullet,\bullet}) \circ \alpha_{\bullet,\bullet,\bullet} \circ (\alpha_{\bullet,\bullet,\bullet} \otimes \mathrm{id}_\bullet)(((\bullet \otimes \bullet) \otimes \bullet) \otimes \bullet) \\
&= (\mathrm{id}_\bullet \otimes \alpha_{\bullet,\bullet,\bullet}) \circ \alpha_{\bullet,\bullet,\bullet}((\bullet \otimes (\bullet \otimes \bullet)) \otimes \bullet) \\
&= (\mathrm{id}_\bullet \otimes \alpha_{\bullet,\bullet,\bullet})(\bullet \otimes ((\bullet \otimes \bullet) \otimes \bullet)) \\
&= (\bullet \otimes (\bullet \otimes (\bullet \otimes \bullet))) \\
&= \bullet \otimes (\bullet \otimes \bullet) \\
&= \bullet \otimes \bullet \\
&= \bullet.
\end{aligned}$$

And the right-hand side of the pentagon identity is also

$$\begin{aligned}
\alpha_{\bullet,\bullet,\bullet} \circ \alpha_{\bullet,\bullet,\bullet}(((\bullet \otimes \bullet) \otimes \bullet) \otimes \bullet) &= \alpha_{\bullet,\bullet,\bullet}((\bullet \otimes \bullet) \otimes (\bullet \otimes \bullet)) \\
&= \bullet \otimes (\bullet \otimes (\bullet \otimes \bullet)) \\
&= \bullet \otimes (\bullet \otimes \bullet) \\
&= \bullet \otimes \bullet \\
&= \bullet.
\end{aligned}$$

(iv) Since $\bullet$ is the only object of $\mathcal{C}$, then the only pair of objects of $\mathcal{C}$ is $(\bullet, \bullet)$. Moreover, since $\mathbf{1} = \bullet$ and $\lambda_\bullet = \alpha_{\bullet,\bullet,\bullet} = \varrho_\bullet = \mathrm{id}_\bullet$, then the left-hand side of the triangle identity is

$$\begin{aligned}(\mathrm{id}_\bullet \otimes \lambda_\bullet) \circ \alpha_{\bullet,\bullet,\bullet}((\bullet \otimes \bullet) \otimes \bullet) &= (\mathrm{id}_\bullet \otimes \lambda_\bullet)(\bullet \otimes (\bullet \otimes \bullet)) \\ &= \bullet \otimes \bullet \\ &= \bullet.\end{aligned}$$

And the right-hand side of the triangle identity is also

$$\begin{aligned}\varrho_\bullet \circ \alpha_{\bullet,\bullet,\bullet}((\bullet \otimes \bullet) \otimes \bullet) &= \varrho_\bullet(\bullet \otimes (\bullet \otimes \bullet)) \\ &= \varrho_\bullet(\bullet \otimes \bullet) \\ &= \bullet.\end{aligned}$$

This shows that the pair $(\mathcal{C}, \otimes)$ is indeed a monoidal category.

As mentioned in the previous section, products also induce monoidal structures in the category of small categories.

Example 3.4.3. Recall from Proposition 3.3.3 that products induce functors on categories that admit finite products. Moreover, this functor satisfies conditions (i)-(iv) of Definition 3.4.1. This means that any category that admits finite products admits a monoidal structure. Similarly, one can verify that any category that admits finite coproducts also admits a monoidal structure.

As we mentioned in the beginning of this section, the functor $\otimes$ in a monoidal category is a generalization of the tensor product of vector spaces. In the next example, we verify that, in fact, the usual tensor product endows the category of vector spaces with a monoidal structure.

Example 3.4.4. Let $\Bbbk$ be a field and let $\mathcal{C}$ denote the category of vector spaces over $\Bbbk$. That is, the objects of $\mathcal{C}$ are the vector spaces over $\Bbbk$, the morphisms of $\mathcal{C}$ are the linear transformations between these vector spaces, and the composition is the usual composition of functions (see Example 1.5.8).

Now, let $\otimes$ be the functor that assigns the usual tensor product $V \otimes_\Bbbk W$ to a pair of $\Bbbk$-vector spaces (V, W) and assigns the linear transformation

$$(T_1 \otimes T_2) : V_1 \otimes_\Bbbk V_2 \to W_1 \otimes_\Bbbk W_2,$$

given by

$$(T_1 \otimes T_2)\left(\sum_{i=1}^{n} \lambda_i\, v_{1,i} \otimes v_{2,i}\right) = \sum_{i=1}^{n} \lambda_i\, T_1(v_{1,i}) \otimes T_2(v_{2,i}),$$

to a pair of linear transformations $(T_1 : V_1 \to W_1,\, T_2 : V_2 \to W_2)$. In order to show that $(\mathcal{C}, \otimes)$ is a monoidal category, we will verify that the conditions (i)-(iv) are satisfied.

(i) Recall (for instance, from [KM97, §4.2.2]) that, for each triple (V_1, V_2, V_3) of $\Bbbk$-vector spaces, there is a linear isomorphism

$$\alpha_{V_1,V_2,V_3} : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3),$$

given explicitly by

$$\alpha_{V_1,V_2,V_3}\left(\sum_{i=1}^{n} \lambda_i\, (v_{1,i} \otimes v_{2,i}) \otimes v_{3,i}\right) = \sum_{i=1}^{n} \lambda_i\, v_{1,i} \otimes (v_{2,i} \otimes v_{3,i}).$$

To verify that α is a natural transformation, let (W_1, W_2, W_3) be another triple of $\Bbbk$-vector spaces and $T_1 : V_1 \to W_1$, $T_2 : V_2 \to W_2$, $T_3 : V_3 \to W_3$ be a triple of linear transformations. The naturality condition is satisfied by α because

$$\begin{aligned} T_1 \otimes (T_2 \otimes T_3) & \left(\alpha_{V_1,V_2,V_3}\left(\sum_{i=1}^{n} \lambda_i (v_{i,1} \otimes v_{2,i}) \otimes v_{3,i}\right)\right) \\ &= T_1 \otimes (T_2 \otimes T_3)\left(\sum_{i=1}^{n} \lambda_i v_{i,1} \otimes (v_{2,i} \otimes v_{3,i})\right) \\ &= \sum_{i=1}^{n} \lambda_i T_1(v_{i,1}) \otimes (T_2(v_{2,i}) \otimes T_3(v_{3,i})) \end{aligned}$$

is equal to

$$\begin{aligned} \alpha_{W_1,W_2,W_3} & \left((T_1 \otimes T_2) \otimes T_3 \left(\sum_{i=1}^{n} \lambda_i (v_{i,1} \otimes v_{2,i}) \otimes v_{3,i}\right)\right) \\ &= \alpha_{W_1,W_2,W_3}\left(\sum_{i=1}^{n} \lambda_i (T_1(v_{i,1}) \otimes T_2(v_{2,i})) \otimes T_3(v_{3,i})\right) \\ &= \sum_{i=1}^{n} \lambda_i T_1(v_{i,1}) \otimes (T_2(v_{2,i}) \otimes T_3(v_{3,i})), \end{aligned}$$

for all $n \geq 0$, $\lambda_1, \ldots, \lambda_n \in \Bbbk$, $v_{1,1}, \ldots, v_{1,n} \in V_1$, $v_{2,1}, \ldots, v_{2,n} \in V_2$ and $v_{3,1}, \ldots, v_{3,n} \in V_3$. This shows that α is in fact a natural isomorphism.

(ii) Define the identity object $\mathbf{1}$ as the 1-dimensional vector space $\Bbbk$. Then, recall (for instance, from [KM97, §4.1.7]) that, for each $\Bbbk$-vector space V, there is a linear isomorphism $\lambda_V : \Bbbk \otimes_\Bbbk V \to V$, explicitly given by

$$\lambda_V \left(\sum_{i=1}^n \alpha_i \otimes v_i \right) = \sum_{i=1}^n \alpha_i v_i.$$

To verify that λ is also a natural transformation, let W be another $\Bbbk$-vector space and $T : V \to W$ be a linear transformation. The naturality condition is satisfied by λ because

$$T \left(\lambda_V \left(\sum_{i=1}^n \alpha_i \otimes v_i \right) \right) = T \left(\sum_{i=1}^n \alpha_i v_i \right) = \sum_{i=1}^n \alpha_i T(v_i)$$

is equal to

$$\lambda_W \left(\mathrm{id}_\Bbbk \otimes T \left(\sum_{i=1}^n \alpha_i \otimes v_i \right) \right) = \lambda_W \left(\sum_{i=1}^n \alpha_i \otimes T(v_i) \right) = \sum_{i=1}^n \alpha_i T(v_i),$$

for all $n \geq 0$, $\alpha_1, \ldots, \alpha_n \in \Bbbk$ and $v_1, \ldots, v_n \in V$. This shows that λ is a natural isomorphism.

Next, recall that, for each $\Bbbk$-vector space V, there is also a linear isomorphism $\varrho_V : V \otimes_\Bbbk \Bbbk \to V$, explicitly given by

$$\varrho_V \left(\sum_{i=1}^n v_i \otimes \alpha_i \right) = \sum_{i=1}^n \alpha_i v_i.$$

The verification that ϱ is a natural transformation is very similar to the one shown above for λ.

(iii) Let V_1, V_2, V_3, V_4 be four $\Bbbk$-vector spaces and $v_1 \in V_1$, $v_2 \in V_2$, $v_3 \in V_3$ and $v_4 \in V_4$ be vectors in these vector spaces. In this case, the left side of the pentagon inequality is

$$\begin{aligned}
(\mathrm{id}_{V_1} \otimes \alpha_{V_2,V_3,V_4}) \circ \alpha_{V_1,V_2 \otimes V_3,V_4} \circ (\alpha_{V_1,V_2,V_3} \otimes \mathrm{id}_{V_4})(((v_1 \otimes v_2) \otimes v_3) \otimes v_4) \\
= (\mathrm{id}_{V_1} \otimes \alpha_{V_2,V_3,V_4}) \circ \alpha_{V_1,V_2 \otimes V_3,V_4}((v_1 \otimes (v_2 \otimes v_3)) \otimes v_4) \\
= (\mathrm{id}_{V_1} \otimes \alpha_{V_2,V_3,V_4})(v_1 \otimes ((v_2 \otimes v_3) \otimes v_4)) \\
= v_1 \otimes (v_2 \otimes (v_3 \otimes v_4)).
\end{aligned}$$

And the right side of the pentagon identity is

$$\begin{aligned}\alpha_{V_1,V_2,V_3\otimes V_4}\left(\alpha_{V_1\otimes V_2,V_3,V_4}(((v_1\otimes v_2)\otimes v_3)\otimes v_4)\right) &= \alpha_{V_1,V_2,V_3\otimes V_4}((v_1\otimes v_2)\otimes(v_3\otimes v_4)) \\ &= v_1\otimes(v_2\otimes(v_3\otimes v_4)).\end{aligned}$$

Since these two sides are equal, we conclude that α satisfies the pentagon identity.

(iv) Let V_1, V_2 be a pair of $\Bbbk$-vector spaces, let $v_1 \in V_1$ and $v_2 \in V_2$ be vectors, and let $k \in \Bbbk$ be a scalar. Since the identity object $\mathbf{1}$ is the 1-dimensional vector space $\Bbbk$, the left side of the triangle identity is

$$\begin{aligned}(\mathrm{id}_{V_1}\otimes\lambda_{V_2})\left(\alpha_{V_1,\Bbbk,V_2}((v_1\otimes k)\otimes v_2)\right) &= (\mathrm{id}_{V_1}\otimes\lambda_{V_2})(v_1\otimes(k\otimes v_2)) \\ &= v_1\otimes(kv_2) \\ &= k(v_1\otimes v_2).\end{aligned}$$

And the right side of the right side of the triangle identity is

$$\varrho_{V_1}\otimes\mathrm{id}_{V_2}((v_1\otimes k)\otimes v_2) = (kv_1)\otimes v_2 = k(v_1\otimes v_2).$$

Since these two sides are equal, we conclude that the triangle identity is also satisfied.

This shows that conditions (i)-(iv) are satisfied, and hence that the usual tensor product endows the category of vector spaces with a monoidal structure.

To close this section we will construct a category which admits no monoidal structure. To help us do that, we will prove the following general result.

Proposition 3.4.5. Let $(\mathcal{C}, \otimes)$ be a monoidal category. If we denote its identity object by $\mathbf{1}$, then $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})$ is an abelian monoid with respect to the composition of $\mathcal{C}$.

Proof. We begin by showing that $(\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1}), \circ)$ is a monoid. Indeed, since $\circ$ is the composition of the category $\mathcal{C}$, it is an associative operation. Moreover, since $\mathcal{C}$ is a category, there exists a morphism, $\mathrm{id}_{\mathbf{1}} \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1})$, which satisfies

$$\mathrm{id}_{\mathbf{1}}\circ f = f \quad\text{and}\quad f\circ\mathrm{id}_{\mathbf{1}} = f \qquad \text{for all } f\in\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1}).$$

This shows that $(\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},\mathbf{1}), \circ)$ is a monoid.

Next, we will use the fact that $\lambda_{\mathbf{1}} = \varrho_{\mathbf{1}}$ to show that this monoid is abelian. (The proof of this fact will be given in Lemma 3.4.7.) Since λ is a natural isomorphism, for each pair of morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$, we have

$$f \circ \lambda_{\mathbf{1}} = \lambda_{\mathbf{1}} \circ (\mathrm{id}_{\mathbf{1}} \otimes f) \quad \text{and} \quad g \circ \lambda_{\mathbf{1}} = \lambda_{\mathbf{1}} \circ (\mathrm{id}_{\mathbf{1}} \otimes g),$$

or equivalently,

$$f = \lambda_{\mathbf{1}} \circ (\mathrm{id}_{\mathbf{1}} \otimes f) \circ \lambda_{\mathbf{1}}^{-1} \quad \text{and} \quad g = \lambda_{\mathbf{1}} \circ (\mathrm{id}_{\mathbf{1}} \otimes g) \circ \lambda_{\mathbf{1}}^{-1}.$$

Similarly, since ϱ is also a natural isomorphism, we also have

$$f = \varrho_{\mathbf{1}} \circ (f \otimes \mathrm{id}_{\mathbf{1}}) \circ \varrho_{\mathbf{1}}^{-1} \quad \text{and} \quad g = \varrho_{\mathbf{1}} \circ (g \otimes \mathrm{id}_{\mathbf{1}}) \circ \varrho_{\mathbf{1}}^{-1}.$$

Hence, using the fact that $\lambda_{\mathbf{1}} = \varrho_{\mathbf{1}}$, we obtain that

$$\begin{aligned} f \circ g &= \left(\varrho_{\mathbf{1}} \circ (f \otimes \mathrm{id}_{\mathbf{1}}) \circ \varrho_{\mathbf{1}}^{-1}\right) \circ \left(\lambda_{\mathbf{1}} \circ (\mathrm{id}_{\mathbf{1}} \otimes g) \circ \lambda_{\mathbf{1}}^{-1}\right) \\ &= \varrho_{\mathbf{1}} \circ (f \otimes g) \circ \lambda_{\mathbf{1}}^{-1} \\ &= \lambda_{\mathbf{1}} \circ (f \otimes g) \circ \varrho_{\mathbf{1}}^{-1} \\ &= \left(\lambda_{\mathbf{1}} \circ (\mathrm{id}_{\mathbf{1}} \otimes g) \circ \lambda_{\mathbf{1}}^{-1}\right) \circ \left(\varrho_{\mathbf{1}} \circ (f \otimes \mathrm{id}_{\mathbf{1}}) \circ \varrho_{\mathbf{1}}^{-1}\right) \\ &= g \circ f. \end{aligned}$$

□

Using the result above, we can now easily construct a category which admits no monoidal structure.

Example 3.4.6. Let S_3 denote the group of permutations of a set with three elements. Recall that this is a non-abelian group and, in particular, a non-abelian monoid. If we consider a category $\mathcal{C}$ with one object, $\mathrm{Obj}(\mathcal{C}) = \{\bullet\}$, six morphisms, $\mathrm{Mor}(\mathcal{C}) = \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) = S_3$, and composition induced by the composition of permutations in S_3, according to Proposition 3.4.5, this category admits no monoidal structure.

To formally complete the proof of Proposition 3.4.5, we will prove the following technical result.

Lemma 3.4.7. Let $(\mathcal{C}, \otimes)$ be a monoidal category. If we denote by $\mathbf{1}$ its identity object and by λ, ϱ the natural isomorphisms

$$\{\lambda_x : \mathbf{1} \otimes x \to x \mid x \in \mathrm{Obj}(\mathcal{C})\} \quad \text{and} \quad \{\varrho_x : x \otimes \mathbf{1} \to x \mid x \in \mathrm{Obj}(\mathcal{C})\},$$

then we have $\lambda_{\mathbf{1}} = \varrho_{\mathbf{1}}$.

Proof. We will begin by proving that

$$\mathrm{id}_{\mathbf{1}} \otimes (\lambda_x \otimes \mathrm{id}_y) = (\mathrm{id}_{\mathbf{1}} \otimes \lambda_{x \otimes y}) \circ (\mathrm{id}_{\mathbf{1}} \otimes \alpha_{\mathbf{1},x,y})$$

for every pair of objects x, y of $\mathcal{C}$. In fact, using the pentagon identity for $a = b = \mathbf{1}$, $c = x$ and $d = y$, the triangle identity and the naturality of α, we see that

$$\begin{aligned}
&(\mathrm{id}_{\mathbf{1}} \otimes \lambda_{x\otimes y}) \circ (\mathrm{id}_{\mathbf{1}} \otimes \alpha_{\mathbf{1},x,y}) \\
&\qquad = (\mathrm{id}_{\mathbf{1}} \otimes \lambda_{x\otimes y}) \circ \alpha_{\mathbf{1},\mathbf{1},x\otimes y} \circ \alpha_{\mathbf{1}\otimes\mathbf{1},x,y} \circ (\alpha_{\mathbf{1},\mathbf{1},x} \otimes \mathrm{id}_y)^{-1} \circ \alpha^{-1}_{\mathbf{1},\mathbf{1}\otimes x,y} \\
&\qquad = (\varrho_{\mathbf{1}} \otimes \mathrm{id}_{x\otimes y}) \circ \alpha_{\mathbf{1}\otimes\mathbf{1},x,y} \circ (\alpha_{\mathbf{1},\mathbf{1},x} \otimes \mathrm{id}_y)^{-1} \circ \alpha^{-1}_{\mathbf{1},\mathbf{1}\otimes x,y} \\
&\qquad = \alpha_{\mathbf{1},x,y} \circ ((\varrho_{\mathbf{1}} \otimes \mathrm{id}_x) \otimes \mathrm{id}_y) \circ (\alpha_{\mathbf{1},\mathbf{1},x} \otimes \mathrm{id}_y)^{-1} \circ \alpha^{-1}_{\mathbf{1},\mathbf{1}\otimes x,y} \\
&\qquad = \alpha_{\mathbf{1},x,y} \circ ((\mathrm{id}_{\mathbf{1}} \otimes \lambda_x) \otimes \mathrm{id}_y) \circ \alpha^{-1}_{\mathbf{1},\mathbf{1}\otimes x,y} \\
&\qquad = \mathrm{id}_{\mathbf{1}} \otimes (\lambda_x \otimes \mathrm{id}_y) .
\end{aligned}$$

Now, we will use the identity proved in the previous paragraph to prove that $\lambda_x \otimes \mathrm{id}_y = \lambda_{x\otimes y} \circ \alpha_{\mathbf{1},x,y}$ for every pair of objects x, y of $\mathcal{C}$. In fact, using the naturality of λ and α, we see that

$$\begin{aligned}
\lambda_x \otimes \mathrm{id}_y &= \lambda_{x\otimes y} \circ (\mathrm{id}_{\mathbf{1}} \otimes (\lambda_x \otimes \mathrm{id}_y)) \circ \lambda^{-1}_{(1\otimes x)\otimes y} \\
&= \lambda_{x\otimes y} \circ (\mathrm{id}_{\mathbf{1}} \otimes (\lambda_{x\otimes y} \circ \alpha_{\mathbf{1},x,y})) \circ \lambda^{-1}_{(1\otimes x)\otimes y} \\
&= \lambda_{x\otimes y} \circ \alpha_{\mathbf{1},x,y}.
\end{aligned}$$

Now, we will use the identity $\lambda_x \otimes \mathrm{id}_y = \lambda_{x\otimes y} \circ \alpha_{\mathbf{1},x,y}$ to show that

$$\lambda_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} = \varrho_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} .$$

In fact, if we choose $x = y = \mathbf{1}$, the identity $\lambda_x \otimes \mathrm{id}_y = \lambda_{x\otimes y} \circ \alpha_{\mathbf{1},x,y}$ becomes

$$\lambda_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} = \lambda_{\mathbf{1}\otimes\mathbf{1}} \circ \alpha_{\mathbf{1},\mathbf{1},\mathbf{1}}.$$

Using the naturality of λ, we obtain that $\lambda_{\mathbf{1}\otimes\mathbf{1}} = \mathrm{id}_{\mathbf{1}} \otimes \lambda_{\mathbf{1}}$, which implies that the identity above becomes

$$\lambda_{\nVdash} \otimes \mathrm{id}_{\mathbf{1}} = (\mathrm{id}_{\mathbf{1}} \otimes \lambda_{\mathbf{1}}) \circ \alpha_{\mathbf{1},\mathbf{1},\mathbf{1}}.$$

Now, using the triangular identity (Definition 3.4.1 (iv)) with $x = y = \mathbf{1}$, we obtain that

$$\lambda_{\nVdash} \otimes \mathrm{id}_{\mathbf{1}} = \varrho_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} .$$

To conclude this proof, we will use the identity $\lambda_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} = \varrho_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}}$ to show that $\lambda_{\mathbf{1}} = \varrho_{\mathbf{1}}$. In fact, using the naturality of ϱ, we see that

$$\begin{aligned}\varrho_{\mathbf{1}} &= \lambda_{\mathbf{1}} \circ \varrho_{\mathbf{1}\otimes\mathbf{1}} \circ (\lambda_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}})^{-1} \\ &= \lambda_{\mathbf{1}} \circ \varrho_{\mathbf{1}\otimes\mathbf{1}} \circ (\varrho_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}})^{-1} \\ &= \lambda_{\mathbf{1}} \circ \varrho_{\mathbf{1}\otimes\mathbf{1}} \circ \varrho_{\mathbf{1}\otimes\mathbf{1}}^{-1} \\ &= \lambda_{\mathbf{1}}.\end{aligned}$$

□

3.5. Braided and Symmetric Monoidal Categories

Braided and symmetric categories are monoidal categories also equipped with a structure that generalizes the commutativity for tensor products. Formally, this structure is a natural isomorphism that identifies objects that differ by the order in which they are tensored. In this section, we will present the abstract definitions of braided and symmetric monoidal categories and illustrate these definitions with concrete examples.

Definition 3.5.1. A monoidal category $(\mathcal{C}, \otimes)$ is said to be *braided* when it is equipped with a natural isomorphism

$$\{\sigma_{x,y} : x \otimes y \to y \otimes x \mid x, y \in \mathrm{Obj}(\mathcal{C})\}$$

that satisfies the following identities:

(i) $\alpha_{b,c,a} \circ \sigma_{a,\, b\otimes c} \circ \alpha_{a,b,c} = (\mathrm{id}_b \otimes \sigma_{a,c}) \circ \alpha_{b,a,c} \circ (\sigma_{a,b} \otimes \mathrm{id}_c)$,

(ii) $\alpha_{b,c,a} \circ \sigma^{-1}_{b\otimes c,\, a} \circ \alpha_{a,b,c} = (\mathrm{id}_b \otimes \sigma^{-1}_{c,a}) \circ \alpha_{b,a,c} \circ (\sigma^{-1}_{b,a} \otimes \mathrm{id}_c)$.

In this case, the natural isomorphism σ is called *braiding* and the identities (i) and (ii) are called *hexagon identities* (see Figure 3.5.1). Moreover, the category $\mathcal{C}$ is said to be *symmetric* when $\sigma_{y,x} \circ \sigma_{x,y} = \mathrm{id}_{x\otimes y}$ for every pair of objects x, y of $\mathcal{C}$.

To illustrate the abstract definition above, we will consider some concrete examples. We begin with the smallest monoidal category.

Example 3.5.2. Recall from Example 3.4.2 that the smallest category, the category $\mathcal{C}$ with one object, $\mathrm{Obj}(\mathcal{C}) = \{\bullet\}$, one morphism, $\mathrm{Mor}(\mathcal{C}) = \{\mathrm{id}_\bullet\}$, and composition given by $\mathrm{id}_\bullet \circ \mathrm{id}_\bullet = \mathrm{id}_\bullet$, admits a monoidal structure, explicitly given by

$$\bullet \otimes \bullet = \bullet \quad \text{and} \quad \mathrm{id}_\bullet \otimes \mathrm{id}_\bullet = \mathrm{id}_\bullet\,.$$

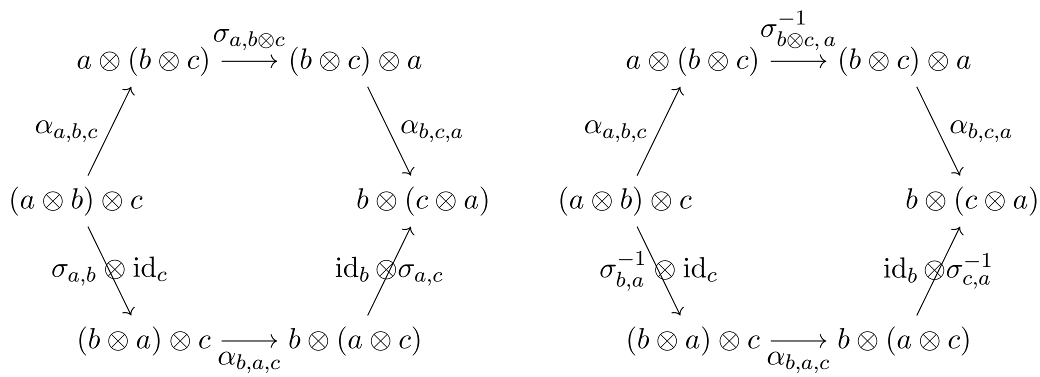


FIGURE 3.5.1. Diagrams of hexagon identities

Since this category has only one object and one morphism, the identity is a braiding on it. In fact, $\mathrm{id}_\bullet$ is an isomorphism $\bullet \otimes \bullet \to \bullet \otimes \bullet$. Moreover, since $\mathrm{id}_\bullet \circ \mathrm{id}_\bullet = \mathrm{id}_\bullet$, this braided monoidal category is also symmetric.

In the next example, we verify that the monoidal category of vector spaces is also braided and symmetric.

Example 3.5.3. Let $\Bbbk$ be a field and $\mathcal{C}$ be the category of vector spaces over $\Bbbk$. Recall from Example 3.4.4 that the usual tensor product endows this category with a monoidal structure. We will show that there exists also a symmetric braiding on $\mathcal{C}$

In fact, recall (for instance, from [KM97, §4.2.3]) that, for every pair of $\Bbbk$-vector spaces V, W, there exists a linear isomorphism $\sigma_{V,W} : V \otimes W \to W \otimes V$, explicitly given by

$$\sigma_{V,W}\left(\sum_{i=1}^{n} \lambda_i (v_i \otimes w_i)\right) = \sum_{i=1}^{n} \lambda_i (w_i \otimes v_i).$$

We will verify that the family $\{\sigma_{V,W} \mid V, W \in \mathrm{Obj}(\mathcal{C})\}$ defines a natural transformation and satisfies the hexagon identities.

To verify that σ is a natural transformation, let $T : V \to V'$, $S : W \to W'$ be a pair of linear transformations. We want to check that

$$(S \otimes T) \circ \sigma_{V,W} = \sigma_{V',W'} \circ (T \otimes S).$$

In fact, observe that

$$\begin{aligned}
(S\otimes T)\left(\sigma_{V,W}\left(\sum_{i=1}^{n}\lambda_i(v_i\otimes w_i)\right)\right) &= S\otimes T\left(\sum_{i=1}^{n}\lambda_i(w_i\otimes v_i)\right)\\
&= \sum_{i=1}^{n}\lambda_i(S(w_i)\otimes T(v_i))\\
&= \sigma_{V',W'}\left(\sum_{i=1}^{n}\lambda_i(T(v_i)\otimes S(w_i))\right)\\
&= \sigma_{V',W'}\left((T\otimes S)\left(\sum_{i=1}^{n}\lambda_i(v_i\otimes w_i)\right)\right),
\end{aligned}$$

for all $\lambda_1,\dots,\lambda_n\in\Bbbk$, $v_1,\dots,v_n\in V$ and $w_1,\dots,w_n\in W$. This shows that σ defines a natural transformation, and thus, a natural isomorphism.

Next, we will verify that σ satisfies the hexagon identities. To do that, fix a triple of $\Bbbk$-vector spaces, V,W,U. Then, identity (i) in Definition 3.5.1 follows from the fact that

$$\begin{aligned}
\alpha_{b,c,a}\left(\sigma_{a,\,b\otimes c}\left(\alpha_{a,b,c}((v\otimes w)\otimes u)\right)\right) &= \alpha_{b,c,a}\left(\sigma_{a,\,b\otimes c}(v\otimes(w\otimes u))\right)\\
&= \alpha_{b,c,a}((w\otimes u)\otimes v)\\
&= w\otimes(u\otimes v)
\end{aligned}$$

is equal to

$$\begin{aligned}
(\mathrm{id}_b\otimes\sigma_{a,c})\left(\alpha_{b,a,c}\left((\sigma_{a,b}\otimes\mathrm{id}_c)((v\otimes w)\otimes u)\right)\right) &= (\mathrm{id}_b\otimes\sigma_{a,c})\left(\alpha_{b,a,c}((w\otimes v)\otimes u)\right)\\
&= (\mathrm{id}_b\otimes\sigma_{a,c})(w\otimes(v\otimes u))\\
&= w\otimes(u\otimes v),
\end{aligned}$$

for all $v\in V$, $w\in W$ and $u\in U$. Similarly, identity (ii) in Definition 3.5.1 follows from the fact that

$$\begin{aligned}
\alpha_{b,c,a}\left(\sigma^{-1}_{b\otimes c,a}\left(\alpha_{a,b,c}((v\otimes w)\otimes u)\right)\right) &= \alpha_{b,c,a}\left(\sigma^{-1}_{b\otimes c,a}(v\otimes(w\otimes u))\right)\\
&= \alpha_{b,c,a}((w\otimes u)\otimes v)\\
&= w\otimes(u\otimes v)
\end{aligned}$$

is equal to

$$\begin{aligned}(\mathrm{id}_b \otimes \sigma_{c,a}^{-1})\left(\alpha_{b,a,c}\left((\sigma_{b,a}^{-1} \otimes \mathrm{id}_c)((v \otimes w) \otimes u)\right)\right) \\ &= (\mathrm{id}_b \otimes \sigma_{c,a}^{-1})\left(\alpha_{b,a,c}((w \otimes v) \otimes u)\right) \\ &= (\mathrm{id}_b \otimes \sigma_{c,a}^{-1})(w \otimes (v \otimes u)) \\ &= w \otimes (u \otimes v),\end{aligned}$$

for all $v \in V$, $w \in W$ and $u \in U$.

We close this section by constructing an example of a monoidal category that is not braided.

Example 3.5.4. To construct a monoidal category that is not braided, we will begin by constructing a small non-abelian monoid. In fact, consider the set $M = \{e, a, b\}$, endowed with the operation $\cdot : M \times M \to M$ defined by

$$\begin{aligned} e \cdot e = e, \quad & e \cdot a = a, \quad e \cdot b = b, \\ a \cdot e = a, \quad & a \cdot a = a, \quad a \cdot b = a, \\ b \cdot e = b, \quad & b \cdot a = b, \quad b \cdot b = b. \end{aligned}$$

One can see that e is the identity element in $(M, \cdot)$, and one can explicitly check that the operation $\cdot$ is associative. This means that $(M, \cdot)$ is a monoid.

Now, consider the category $\mathcal{C}$ with three objects, $\mathrm{Obj}(\mathcal{C}) = M$, three morphisms, $\mathrm{Mor}(\mathcal{C}) = \{\mathrm{id}_e, \mathrm{id}_a, \mathrm{id}_b\}$, and the obvious composition,

$$\mathrm{id}_e \circ \mathrm{id}_e = \mathrm{id}_e, \quad \mathrm{id}_a \circ \mathrm{id}_a = \mathrm{id}_a \quad \text{and} \quad \mathrm{id}_b \circ \mathrm{id}_b = \mathrm{id}_b\,.$$

We can endow the category $\mathcal{C}$ with a monoidal structure by defining

$$x \otimes y := x \cdot y \quad \text{and} \quad \mathrm{id}_x \otimes \mathrm{id}_y = \mathrm{id}_{x \cdot y} \qquad \text{for all } x, y \in \mathrm{Obj}(\mathcal{C}).$$

In fact, the identity object $\mathbf{1}$ is e and the natural isomorphisms α, λ, ϱ are all equal to the identity one (see Example 3.1.2).

Now, since $a \otimes b = a \cdot b = a$, $b \otimes a = b \cdot a = b$, and there exists no morphisms in $\mathrm{Hom}_{\mathcal{C}}(a, b)$, the monoidal category $(\mathcal{C}, \otimes)$ cannot be braided.

Part IV

Stratifications

In this part of these notes, we introduce the notion of stratification of abelian categories. This concept provides a way to decompose an abelian category into simpler layers, each of which interacts with the others in a controlled manner. It generalizes familiar decompositions from algebraic geometry and representation theory, such as filtrations by support or by weight, to an abstract categorical framework.

To define stratifications rigorously, we must first develop a few key notions. We begin by recalling the concept of *subcategory*, the categorical analogue of subsets, subspaces and subgroups. Among these, certain subcategories known as *Serre subcategories* play a central role in abelian settings: they are precisely those for which one can construct meaningful quotient categories. The corresponding quotient construction, called *Serre quotient*, allows us to collapse a Serre subcategory while preserving exactness. In turn, this leads naturally to the concept of *recollement*, which formalizes how an abelian category can be reconstructed from a subcategory and its quotient.

Each of these constructions contributes to the definition of stratification: a stratification of an abelian category is, informally, a layered structure built from a finite sequence of recollements indexed by a poset. Hence, the goal of this part is thus twofold: to build the categorical tools necessary for defining stratifications, and to illustrate how these tools mirror well-known constructions in algebraic geometry and homological algebra.

4.1. Subcategories

Just as we study subspaces of vector spaces and subgroups of groups, we can also consider *subcategories* of a given category. A subcategory consists of a selection of objects and morphisms from the ambient category that themselves form a category under the same composition law. In this section, we define subcategories and illustrate this definition with some basic examples.

Definition 4.1.1 (subcategory)**.** Given a category $\mathcal{C}$, a *subcategory* $\mathcal{D}$ of $\mathcal{C}$ consists of a collection of objects $\mathrm{Obj}(\mathcal{D}) \subseteq \mathrm{Obj}(\mathcal{C})$ and a collection of morphisms $\mathrm{Mor}(\mathcal{D}) \subseteq \mathrm{Mor}(\mathcal{C})$ satisfying the following conditions:

(i) For every object $d \in \mathrm{Obj}(\mathcal{D})$, the identity morphism id_d is in $\mathrm{Mor}(\mathcal{D})$,

(ii) For every pair of morphisms, $f, g \in \mathrm{Mor}(\mathcal{D})$, such that $f \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(b, c)$, the objects a, b, c are in $\mathrm{Obj}(\mathcal{D})$ and the morphism $(g \circ_{\mathcal{C}} f)$ is in $\mathrm{Mor}(\mathcal{D})$.

In this case, one denotes $\mathrm{Mor}(\mathcal{D}) \cap \mathrm{Hom}_{\mathcal{C}}(a, b)$ by $\mathrm{Hom}_{\mathcal{D}}(a, b)$, for every pair of objects $a, b \in \mathrm{Obj}(\mathcal{D})$. Moreover, one says that the subcategory $\mathcal{D}$ of $\mathcal{C}$ is *full* when $\mathrm{Hom}_{\mathcal{D}}(a, b) = \mathrm{Hom}_{\mathcal{C}}(a, b)$, for every pair of objects $a, b \in \mathrm{Obj}(\mathcal{D})$.

Notice that, if $\mathcal{D}$ is a subcategory of a category $\mathcal{C}$, then $(\mathrm{Obj}(\mathcal{D}), \mathrm{Mor}(\mathcal{D}), \circ_{\mathcal{C}})$ is also a category. This means that this definition captures the intuitive idea that a subcategory is a substructure that respects the categorical structure of the ambient category. The next examples illustrate this definition, beginning with the simplest cases.

Example 4.1.2. Given a category $\mathcal{C}$, the category $\mathcal{C}$ is a subcategory of itself. At the other extreme, the empty subcategory is the one with no objects and no morphisms.

While the example above is simple, it establishes that subcategories exist in abundance. In the next example, we will construct full subcategories of a category.

Example 4.1.3. Given a category $\mathcal{C}$, for every choice of a subset $\mathrm{Obj}(\mathcal{D})$ of $\mathrm{Obj}(\mathcal{C})$, if we choose $\mathrm{Hom}_{\mathcal{D}}(a, b)$ to be $\mathrm{Hom}_{\mathcal{C}}(a, b)$ for every pair of objects $a, b \in \mathrm{Obj}(\mathcal{D})$, then we obtain a full subcategory of $\mathcal{C}$.

The examples above are general, but examining a small enough category to list all the possibilities will also help to understand the definition of subcategories. We close this section with one such example. It shows that even for a small category, there can be multiple subcategories, some full and some not.

Example 4.1.4. Consider a category $\mathcal{C}$ with two objects, $\mathrm{Obj}(\mathcal{C}) = \{a, b\}$, and three morphisms, $\mathrm{Mor}(\mathcal{C}) = \{\mathrm{id}_a, f, \mathrm{id}_b\}$, where $\{f\} = \mathrm{Hom}_{\mathcal{C}}(a, b)$. The subcategories of $\mathcal{C}$ are:

- The empty subcategory,

- The subcategory with $\mathrm{Obj}(\mathcal{D}) = \{a\}$ and $\mathrm{Mor}(\mathcal{D}) = \{\mathrm{id}_a\}$,
- The subcategory with $\mathrm{Obj}(\mathcal{D}) = \{b\}$ and $\mathrm{Mor}(\mathcal{D}) = \{\mathrm{id}_b\}$,
- The subcategory with $\mathrm{Obj}(\mathcal{D}) = \{a, b\}$ and $\mathrm{Mor}(\mathcal{D}) = \{\mathrm{id}_a, \mathrm{id}_b\}$,
- The entire category $\mathcal{C}$.

Notice that, if we chose $\mathrm{Obj}(\mathcal{D}) = \{a\}$ and $\mathrm{Mor}(\mathcal{D}) = \{\mathrm{id}_a, f\}$, then we would not obtain a subcategory, because $f \in \mathrm{Hom}_{\mathcal{C}}(a, b) \cap \mathrm{Mor}(\mathcal{D})$ and $b \notin \mathrm{Obj}(\mathcal{D})$. Using similar arguments we can show that no other choice of $\mathrm{Obj}(\mathcal{D})$ and $\mathrm{Mor}(\mathcal{D})$ would form a subcategory of $\mathcal{C}$.

The notion of subcategory provides the basic setting for all subsequent constructions in this part of these notes, including Serre subcategories, quotient categories, and recollements.

4.2. Serre Subcategories

In the study of abelian categories, it is natural to consider subcategories that are well-behaved with respect to the abelian structure. Among these, *Serre subcategories* play a distinguished role: they are precisely the subcategories that admit a quotient construction producing another abelian category. In this section, we define Serre subcategories and present a few examples. Intuitively, a Serre subcategory is one that is closed under taking subobjects, quotients, and extensions.

Definition 4.2.1 (Serre subcategory)**.** Given an abelian category $\mathcal{A}$, a full subcategory $\mathcal{S}$ of $\mathcal{A}$ is called a *Serre subcategory* when $\mathrm{Obj}(\mathcal{S}) \neq \emptyset$ and, for every short exact sequence $0 \to a \to b \to c \to 0$ in $\mathcal{A}$, we have: $b \in \mathrm{Obj}(\mathcal{S})$ if and only if $a \in \mathrm{Obj}(\mathcal{S})$ and $c \in \mathrm{Obj}(\mathcal{S})$.

This definition has a clear two-out-of-three flavour: the middle term of a short exact sequence is an object of $\mathcal{S}$ precisely when the outer terms are. To understand this condition better, we begin with the most basic examples.

Example 4.2.2. For every abelian category $\mathcal{A}$, the zero subcategory (that is, the full subcategory $\mathcal{S}$ for which $\mathrm{Obj}(\mathcal{S}) = \{0\}$) and the whole category ($\mathcal{S} = \mathcal{A}$) are Serre subcategories of $\mathcal{A}$.

While the example above is simple, it establishes that every abelian category possesses at least one Serre subcategory. The first non-trivial examples arise from vector spaces.

Example 4.2.3. Let $\Bbbk$ be a field and $\mathcal{A}$ be the abelian category of $\Bbbk$-vector spaces (see Example 1.8.2). Then, let $\mathcal{S}$ be the full subcategory of $\mathcal{A}$ such that $\mathrm{Obj}(\mathcal{S})$ consists of the finite-dimensional $\Bbbk$-vector spaces. To show that $\mathcal{S}$ is a Serre subcategory, notice that, for every short exact sequence of $\Bbbk$-vector spaces $0 \to W \to V \to U \to 0$, we have that: V is finite-dimensional if and only if W and U are finite-dimensional. In fact, this follows from the additivity of dimensions (also known as Rank-Nullity Theorem, in this case), $\dim V = \dim W + \dim U$.

This example shows how Serre subcategories can capture size constraints. Similar constructions appear throughout algebraic geometry, where support conditions play an analogous role.

Example 4.2.4. Let X be a Noetherian scheme and let $\mathcal{A} = \mathrm{Coh}(X)$ be the category of coherent sheaves on X. For any closed subset $Z \subseteq X$, the subcategory

$$\mathcal{S}_Z = \{\mathcal{F} \in \mathrm{Coh}(X) \mid \mathrm{Supp}(\mathcal{F}) \subseteq Z\}$$

is a Serre subcategory. To verify this, suppose

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence of coherent sheaves. Then $\mathrm{Supp}(\mathcal{G}) = \mathrm{Supp}(\mathcal{F}) \cup \mathrm{Supp}(\mathcal{H})$, which shows that $\mathcal{G}$ has support in Z if and only if both $\mathcal{F}$ and $\mathcal{H}$ have support in Z. This construction is fundamental to the theory of perverse sheaves and provides the building blocks for stratifications by support, as we will see in the next section.

The next example shows a case of a subcategory of an abelian which is not a Serre subcategory.

Example 4.2.5. Let $\mathcal{A}$ be the abelian category of abelian groups (see Example 1.8.3) and $\mathcal{B}$ be the full subcategory of $\mathcal{A}$ for which $\mathrm{Obj}(\mathcal{B}) = \{\mathbb{Z}\}$. To show that $\mathcal{B}$ *is not* a Serre subcategory of $\mathcal{A}$, recall from Example 2.3.22 that there exists a short exact sequence in $\mathcal{A}$ of the form $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$. Since $\mathbb{Z} \in \mathrm{Obj}(\mathcal{B})$ and $\mathbb{Z}/2\mathbb{Z} \notin \mathrm{Obj}(\mathcal{B})$, this implies that $\mathcal{B}$ is not a Serre subcategory of $\mathcal{A}$.

We will close this series of examples with an important example that will be used in the subsequent section, that of the kernel subcategory.

Example 4.2.6. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. The *kernel* of F, denoted $\operatorname{Ker}(F)$, is defined to be the full subcategory of $\mathcal{A}$ for which $\operatorname{Obj}(\operatorname{Ker}(F)) = \{a \in \mathcal{A} \mid F(a) = 0\}$. To show that $\operatorname{Ker}(F)$ is a Serre subcategory of $\mathcal{A}$, let $0 \to a \to b \to c \to 0$ be a short exact sequence in $\mathcal{A}$ and recall from Proposition 2.4.6 that $0 \to F(a) \to F(b) \to F(c) \to 0$ is a short exact sequence in $\mathcal{B}$.

Now, on the one hand, assume that $F(b) = 0$. In this case, the exactness of the sequence $0 \to F(a) \to 0 \to F(c) \to 0$ at $F(a)$ implies that $F(a) = 0$ and the exactness of this sequence at $F(c)$ implies that $F(c) = 0$. On the other hand, suppose that $F(a) = F(c) = 0$. In this case, the exactness of the sequence $0 \to 0 \to F(b) \to 0 \to 0$ at $F(b)$ implies that $F(b) = 0$. This shows that $\operatorname{Ker}(F)$ is indeed a Serre subcategory of $\mathcal{A}$, a fact that will be used in the next section.

We close this section on Serre subcategories by showing that every Serre subcategory of an abelian category is also abelian.

Proposition 4.2.7. Let $\mathcal{A}$ be an abelian category. If $\mathcal{S}$ is a Serre subcategory of $\mathcal{A}$, then $\mathcal{S}$ is also an abelian category.

Proof. Recall that an abelian category is a pre-additive, additive, pre-abelian category, in which every monomorphism is the kernel and every epimorphism is the cokernel of a morphism. We will successively verify that, when $\mathcal{S}$ is a Serre subcategory of an abelian category, it has each one of these properties.

To verify that $\mathcal{S}$ is a pre-additive category, recall that it is a full subcategory of $\mathcal{A}$. This means that $\operatorname{Hom}_{\mathcal{S}}(a, b) = \operatorname{Hom}_{\mathcal{A}}(a, b)$ for all $a, b \in \operatorname{Obj}(\mathcal{S})$. Since $\mathcal{A}$ is assumed to be an abelian (and, in particular, pre-additive) category, we see that every $\operatorname{Hom}_{\mathcal{S}}(a, b)$ is an abelian group when endowed with the group structure of $\operatorname{Hom}_{\mathcal{A}}(a, b)$.

To verify that $\mathcal{S}$ is an additive category, we will show that $\mathcal{S}$ has an initial object and finite products. To verify that $\mathcal{S}$ has a zero object (which is both initial and terminal), first recall that $\operatorname{Obj}(\mathcal{S})$ is non-empty. Hence we can fix an object $a \in \operatorname{Obj}(\mathcal{S})$. Then recall from Example 2.3.23 that $0 \to a \xrightarrow{\mathrm{id}_a} a \to 0 \to 0$ is a short exact sequence in $\mathcal{A}$. Since $a \in \operatorname{Obj}(\mathcal{S})$ and $\mathcal{S}$ is a Serre subcategory of $\mathcal{A}$, we obtain that $0 \in \operatorname{Obj}(\mathcal{S})$ as well. Finally, since 0 is an initial (and terminal) object of $\mathcal{A}$, we conclude that 0 is an initial object of $\mathcal{S}$.

To verify that every pair of objects $a, b \in \mathrm{Obj}(\mathcal{S})$ has a product in $\mathcal{S}$, recall that, since $\mathcal{A}$ is an abelian category (and, in particular, additive), the product $a \times b$ exists in $\mathcal{A}$. Moreover, recall from Example 2.3.24 that there is a short exact sequence $0 \to a \to (a \times b) \to b \to 0$ in $\mathcal{A}$. Since $a, b \in \mathrm{Obj}(\mathcal{S})$ and $\mathcal{S}$ is a Serre subcategory of $\mathcal{A}$ (by hypothesis), we conclude that $a \times b \in \mathrm{Obj}(\mathcal{S})$. This shows that $\mathcal{S}$ is an additive category.

To verify that $\mathcal{S}$ is a pre-abelian category, we will verify that every morphism in $\mathcal{S}$ has a kernel and cokernel in $\mathcal{S}$. To do that, let $f \in \mathrm{Hom}_{\mathcal{S}}(a, b)$ be a morphism in $\mathcal{S}$. Since f is also a morphism in $\mathcal{A}$ and $\mathcal{A}$ is an abelian category (by hypothesis), the kernel and cokernel of f are in $\mathcal{A}$. Then, recall from Example 2.3.25 that there is a short exact sequence $0 \to \ker(f) \to a \to \mathrm{im}(f) \to 0$ in $\mathcal{A}$. Since $a \in \mathrm{Obj}(\mathcal{S})$ and $\mathcal{S}$ is a Serre subcategory of $\mathcal{A}$ (by hypothesis), this implies that $\ker(f)$ and $\mathrm{im}(f)$ are also in $\mathcal{S}$.

To finish this proof, we verify that every monomorphism is the kernel and every epimorphism is the cokernel of a morphism in $\mathcal{S}$. This follows from the fact that the kernels and cokernels of morphisms in $\mathcal{S}$ are in $\mathcal{S}$ and the hypothesis that $\mathcal{A}$ is an abelian category. $\square$

In the next section we will see how Serre subcategories allow us to form *quotient categories*, extending this idea further.

4.3. Serre Quotients

In many areas of mathematics, one often simplifies a structure by identifying or quotienting out certain substructures. Since Serre subcategories behave well with respect to the abelian structure, they are precisely the subcategories that admit such a quotient construction. In this section, we formalise this process in the context of abelian categories by defining their quotients by Serre subcategories. First, we present the abstract definition, then illustrate it with examples, and finally describe its explicit construction.

Definition 4.3.1 (Serre quotient)**.** Given an abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{S}$, the *quotient category* $\mathcal{A}/\mathcal{S}$ is an abelian category satisfying the following universal property:

- There exists an exact functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{S}$, such that $Q(s) = 0$ for all $s \in \mathrm{Obj}(\mathcal{S})$,

- If $\mathcal{B}$ is an abelian category and there exists an exact functor $F : \mathcal{A} \to \mathcal{B}$ such that $F(s) = 0$ for all $s \in \mathrm{Obj}(\mathcal{S})$, then there exists a unique functor $\overline{F} : \mathcal{A}/\mathcal{S} \to \mathcal{B}$ such that $\overline{F} \circ Q = F$.

This definition, while technical, characterizes the quotient category by a universal property. The intuition behind it is that $\mathcal{A}/\mathcal{S}$ is the largest abelian category obtained from $\mathcal{A}$ by making all objects in $\mathcal{S}$ become zero, while preserving exactness. To help understand this definition, we begin with the most basic case.

Example 4.3.2. Let $\mathcal{A}$ be any abelian category and let $\mathcal{S}$ be the full subcategory of $\mathcal{A}$ such that $\mathrm{Obj}(\mathcal{S}) = \{0\}$. In this case, the quotient category $\mathcal{A}/\mathcal{S}$ is equivalent to $\mathcal{A}$ itself. Indeed, let the functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{S}$ be the identity functor on $\mathcal{A}$. Recall from Example 2.4.2 that $Q = \mathrm{Id}_{\mathcal{A}}$ is exact. Then, notice that $Q(0) = 0$. Next, let $\mathcal{B}$ be an abelian category and $F : \mathcal{A} \to \mathcal{B}$ is any exact functor such that $F(0) = 0$. Finally, notice that a functor $\overline{F} : \mathcal{A} \to \mathcal{B}$ is such that $F = \overline{F} \circ Q = \overline{F} \circ \mathrm{Id}_{\mathcal{A}} = \overline{F}$ if and only if $\overline{F} = F$. This confirms that $\mathcal{A}$ satisfies the universal property for the quotient $\mathcal{A}/\mathcal{S}$.

While the quotient by zero changes nothing, the quotient of a category by itself kills everything. This opposite extreme illustrates how the Serre quotient can collapse an entire category.

Example 4.3.3. Let $\mathcal{A}$ be any abelian category and $\mathcal{S}$ be the $\mathcal{A}$ category itself. In this case, the quotient category $\mathcal{A}/\mathcal{S}$ is equivalent to the zero category (that is, the category with a single object 0 and only the morphism id_0). Indeed, let $Q : \mathcal{A} \to \mathcal{A}/\mathcal{S}$ be the functor that sends every object to 0 and every morphism to id_0. Notice that Q is an exact functor, since $0 \to 0 \to 0 \to 0 \to 0$ is an exact sequence. Next, let $\mathcal{B}$ be an abelian category and $F : \mathcal{A} \to \mathcal{B}$ be an exact functor such that $F(a) = 0$ for all $a \in \mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(\mathcal{S})$. The unique functor $\overline{F} : \mathcal{A}/\mathcal{S} \to \mathcal{B}$ (which sends 0 to 0 and id_0 to id_0) satisfies the equation $\overline{F} \circ Q = F$. This shows that the Serre quotient by the entire category is trivial.

Having seen some examples above, we now turn to the explicit description of Serre quotients. The following result provides a concrete construction of the quotient category. To state this result, we will introduce the following notation. When x and x' are objects of a category and there exists a monomorphism $x \to x'$, we will say that *x is a subobject of x'* and denote it by $x \subseteq x'$. Further, when x is a subobject of x', that is, when there exists a monomorphism

$i : x \to x'$, we will denote the cokernel of this monomorphism by x'/x and call it the *quotient of* x' *by* x.

Theorem 4.3.4. Let $\mathcal{A}$ be a small abelian category and $\mathcal{S}$ be a Serre subcategory.

(a) The relation defined by $(a, b) \leq (a', b')$ if and only if $a' \subseteq a$ and $b \subseteq b'$ is a pre-order on the set $\mathrm{Obj}(\mathcal{A}) \times \mathrm{Obj}(\mathcal{A})$.

(b) For each pair of objects $a, b \in \mathrm{Obj}(\mathcal{A})$, the set $I(a, b)$ defined by

$$\{(x, y) \in \mathrm{Obj}(\mathcal{A}) \times \mathrm{Obj}(\mathcal{A}) \mid x \subseteq a,\ a/x \in \mathrm{Obj}(\mathcal{S}), y \subseteq b \text{ and } y \in \mathrm{Obj}(\mathcal{S})\}$$

is a directed set when endowed with the pre-order $\leq$ induced from the set $\mathrm{Obj}(\mathcal{A}) \times \mathrm{Obj}(\mathcal{A})$.

(c) The triple $(\mathrm{Obj}(\mathcal{Q}), \mathrm{Mor}(\mathcal{Q}), \circ_{\mathcal{Q}})$ given by

$$\mathrm{Obj}(\mathcal{Q}) = \mathrm{Obj}(\mathcal{A}), \quad \mathrm{Hom}_{\mathcal{Q}}(a, b) = \mathrm{colim}_{I(a,b)} \mathrm{Hom}_{\mathcal{A}}(x, y),$$

and $\circ_{\mathcal{Q}}$ induced from $\circ_{\mathcal{A}}$ via the colimit forms a category.

(d) The category $\mathcal{Q} = (\mathrm{Obj}(\mathcal{Q}), \mathrm{Mor}(\mathcal{Q}), \circ_{\mathcal{Q}})$ is abelian.

(e) The functor $Q : \mathcal{A} \to \mathcal{Q}$ defined by

$$Q(a) = a \quad \text{and} \quad Q(f) = \mathrm{colim}_{I(a,b)}(f),$$

for all $a, b \in \mathrm{Obj}(\mathcal{A})$ and $f \in \mathrm{Hom}_{\mathcal{A}}(a, b)$, is exact and moreover $Q(s) = 0$ for all $s \in \mathrm{Obj}(\mathcal{S})$.

(f) For every abelian category $\mathcal{B}$ and every exact functor $F : \mathcal{A} \to \mathcal{B}$ such that $F(s) = 0$ for all $s \in \mathcal{S}$, there exists a unique functor $\overline{F} : \mathcal{Q} \to \mathcal{B}$ such that $\overline{F} \circ Q = F$.

This theorem provides an explicit construction for the quotient category. This construction using colimits of morphisms is technical but makes the quotient category computable in examples. The proof of this result is also technical and can be found in [Gab62, III.1]. Instead of providing a proof for it, we will use it to explicitly describe a quotient of the category of vector spaces.

Example 4.3.5. Let $\Bbbk$ be a field, $\mathcal{A}$ be the abelian category of $\Bbbk$-vector spaces, and $\mathcal{S}$ be the Serre subcategory of $\mathcal{A}$ consisting of finite-dimensional $\Bbbk$-vector spaces. The quotient category $\mathcal{A}/\mathcal{S}$ can be constructed as follows. The objects of $\mathcal{A}/\mathcal{S}$ are $\Bbbk$-vector spaces and the morphisms are linear transformations. Two vector spaces V and W are isomorphic is $\mathcal{A}/\mathcal{S}$ if and only if they are isomorphic modulo finite-dimensional subspaces. (In particular, any two finite-dimensional $\Bbbk$-vector spaces are isomorphic in $\mathcal{A}/\mathcal{S}$.) And two linear transformations $T, S : V \to W$ represent the same morphism in $\mathcal{A}/\mathcal{S}$ if they differ by a map that factors through finite-dimensional spaces. Intuitively, $\mathcal{A}/\mathcal{S}$ records only the infinite-dimensional part of each vector space.

This example shows how Serre quotients provide a way to collapse a subcategory. In the next section, we will see how such quotients fit into the broader framework of *recollements*, which describe how an abelian category can be assembled from a subcategory and its quotient.

4.4. Recollements

After constructing Serre quotients, it is natural to ask how an abelian category can be reconstructed from a subcategory and its quotient. The notion of *recollement* (French for "gluing") answers this question. This notion formalizes the idea of decomposing a category into simpler pieces that fit together coherently. Introduced by Beilinson, Bernstein, and Deligne in their work on perverse sheaves, recollements provide a framework for understanding how a category can be reconstructed from a subcategory and a quotient category. In this section, we define recollements of abelian categories and illustrate this concept through several examples.

Definition 4.4.1 (recollement)**.** A *recollement* of an abelian category $\mathcal{A}$ is a diagram of abelian categories and functors of the form

$$\mathcal{A}' \;\underset{\xleftarrow{\;i^*\;}}{\overset{\xleftarrow{\;i^!\;}}{\xrightarrow{\;i_*\;}}}\; \mathcal{A} \;\underset{\xleftarrow{\;j_!\;}}{\overset{\xleftarrow{\;j_*\;}}{\xrightarrow{\;j^*\;}}}\; \mathcal{A}'',$$

such that:

(i) $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples,

(ii) the functors i_*, $j_!$ and j_* are fully faithful,

(iii) $i^* \circ j_! = 0$,

(iv) for every object $A \in \mathcal{A}$, there exist natural exact sequences

$$j_!j^*A \to A \to i_*i^*A \to 0 \quad \text{and} \quad 0 \to i_*i^!A \to A \to j_*j^*A.$$

In this case, we say that $\mathcal{A}$ is a recollement of $\mathcal{A}'$ and $\mathcal{A}''$.

Intuitively, a recollement is a generalized abstract splitting property in an exact sequence of categories and functors. The splitting on the left not necessarily being equal to the splitting on the right. Hence, this definition encodes a rich structure with many consequences. To build intuition about it, we begin with the most elementary case of recollement, where the decomposition is as simple as possible.

Example 4.4.2. Given an abelian category $\mathcal{A}$, if we choose $\mathcal{A}' = \mathcal{A}$ and $\mathcal{A}'' = 0$, we obtain $\mathcal{A}$ as the following recollement of $\mathcal{A}'$ and $\mathcal{A}''$:

$$\mathcal{A} \begin{matrix} \xleftarrow{\mathrm{Id}_{\mathcal{A}}} \\ \xrightarrow{\mathrm{Id}_{\mathcal{A}}} \\ \xleftarrow[\mathrm{Id}_{\mathcal{A}}]{} \end{matrix} \mathcal{A} \begin{matrix} \xleftarrow{0} \\ \xrightarrow{0} \\ \xleftarrow[0]{} \end{matrix} 0.$$

To justify this claim, we will verify that the diagram above satisfies conditions (i)-(iv) of Definition 4.4.1.

(i) To show that $(\mathrm{Id}_{\mathcal{A}}, \mathrm{Id}_{\mathcal{A}}, \mathrm{Id}_{\mathcal{A}})$ is an adjoint triple, notice that

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{Id}_{\mathcal{A}}(a), b) = \mathrm{Hom}_{\mathcal{A}}(a, b) = \mathrm{Hom}_{\mathcal{A}}(a, \mathrm{Id}_{\mathcal{A}}(b)),$$

for all $a, b \in \mathrm{Obj}(\mathcal{A})$, and to show that $(0, 0, 0)$ is also an adjoint triple, notice that

$$\mathrm{Hom}_{\mathcal{A}}(0(a), b) = \mathrm{Hom}_{\mathcal{A}}(0, b) = \{0\} = \mathrm{Hom}_{\mathcal{A}}(a, 0) = \mathrm{Hom}_{\mathcal{A}}(a, 0(b)),$$

for all $a, b \in \mathrm{Obj}(\mathcal{A})$.

(ii) To show that $\mathrm{Id}_{\mathcal{A}}$ and 0 are fully faithfull, first recall from Example 2.2.2 that $\mathrm{Id}_{\mathcal{A}}$ is a fully faithful functor. The fact that $0 : 0 \to \mathcal{A}$ is also fully faithful follows from the fact that

$$\mathrm{Hom}_0(0, 0) = \{0\} = \mathrm{Hom}_{\mathcal{A}}(0, 0).$$

(iii) The fact that $\mathrm{Id}_{\mathcal{A}} \circ 0 = 0 \circ \mathrm{Id}_{\mathcal{A}} = 0$ follows direct from the definition of 0.

(iv) The exactness of the sequences

$$0(a) \to a \to \mathrm{Id}_{\mathcal{A}}(a) \to 0 \quad \text{and} \quad 0 \to \mathrm{Id}_{\mathcal{A}}(a) \to a \to 0(a),$$

for all $a \in \mathrm{Obj}(\mathcal{A})$, has been proved in Example 2.3.23.

This shows that, in fact $\mathcal{A}$ is a recollement of $\mathcal{A}$ and 0. Similarly, we could also take $\mathcal{A}' = 0$ and $\mathcal{A}'' = \mathcal{A}$ to obtain a similar recollement. Although trivial, these recollements show that every abelian category admits at least one such decomposition.

The trivial recollements above provide no actual decomposition. The first meaningful example arise when we consider the direct sum of two abelian categories.

Example 4.4.3. Given two abelian categories, $\mathcal{A}$ and $\mathcal{B}$, consider their direct sum, that is, consider the category $\mathcal{A} \oplus \mathcal{B}$ whose:

- objects are pairs (a, b) of objects $a \in \mathrm{Obj}(\mathcal{A})$ and $b \in \mathrm{Obj}(\mathcal{B})$,
- morphisms are pairs (f, g) of morphisms $f \in \mathrm{Mor}(\mathcal{A})$ and $g \in \mathrm{Mor}(\mathcal{B})$,
- compositions is induced by the compositions of $\mathcal{A}$ and $\mathcal{B}$ in the following way: if $f_1 \in \mathrm{Hom}_{\mathcal{A}}(a_1, a_2)$, $g_1 \in \mathrm{Hom}_{\mathcal{B}}(b_1, b_2)$, $f_2 \in \mathrm{Hom}_{\mathcal{A}}(a_2, a_3)$ and $g_2 \in \mathrm{Hom}_{\mathcal{B}}(b_2, b_3)$, then
$$(f_2, g_2) \circ_{\mathcal{A}\oplus\mathcal{B}} (f_1, g_1) := (f_2 \circ_{\mathcal{A}} f_1,\ g_2 \circ_{\mathcal{B}} g_1)\,.$$

We can characterize $\mathcal{A} \oplus \mathcal{B}$ as a recollement of $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{A} \overset{i^!}{\underset{i^*}{\leftleftarrows}} \!\!\xrightarrow{\ i_*\ }\! \mathcal{A} \oplus \mathcal{B} \overset{j_*}{\underset{j_!}{\leftleftarrows}} \!\!\xrightarrow{\ j^*\ }\! \mathcal{B}.$$

To justify this claim, we will construct explicit functors $i^!, i_*, i^*, j_*, j^*, j_!$. First, let $i_* : \mathcal{A} \to \mathcal{A} \oplus \mathcal{B}$ be the functor defined by assigning

- the object $(a,\, 0_{\mathcal{B}})$ to an object $a \in \mathrm{Obj}(\mathcal{A})$,
- the morphism $(f,\, 0_{\mathcal{B}})$ to a morphism $f \in \mathrm{Mor}(\mathcal{A})$.

Then, let $i^! = i^* : \mathcal{A} \oplus \mathcal{B} \to \mathcal{A}$ be the functor defined by assigning

- the object a to an object $(a, b) \in \mathrm{Obj}(\mathcal{A} \oplus \mathcal{B})$,
- the morphism f to a morphism $(f, g) \in \mathrm{Mor}(\mathcal{A} \oplus \mathcal{B})$.

Similarly, let $j^* : \mathcal{A} \oplus \mathcal{B} \to \mathcal{B}$ be the functor defined by assigning

- the object b to an object $(a, b) \in \mathrm{Obj}(\mathcal{A} \oplus \mathcal{B})$,
- the morphism g to a morphism $(f, g) \in \mathrm{Mor}(\mathcal{A} \oplus \mathcal{B})$.

Finally, let $j_* = j_! : \mathcal{B} \to \mathcal{A} \oplus \mathcal{B}$ be the functor defined by assigning

- the object $(0_{\mathcal{A}},\, b)$ to an object $b \in \mathrm{Obj}(\mathcal{B})$,
- the morphism $(0_{\mathcal{A}},\, g)$ to a morphism $g \in \mathrm{Mor}(\mathcal{B})$.

The verification that these functors $i^!, i_*, i^*, j_*, j^*, j_!$ actually satisfy conditions (i)-(iv) of Definition 4.4.1 is straight-forward.

The example above is important for its generality. We close this section with an example that illustrates a natural geometric instance of this phenomenon.

Example 4.4.4. Let X be a Noetherian scheme, Z be a closed subscheme of X, and $U = X \setminus Z$ be its open complement. Denote by $i : Z \to X$ and $j : U \to X$ the inclusion maps. Consider the categories $\mathcal{A} = \mathrm{Coh}(X)$, $\mathcal{A}' = \mathrm{Coh}(Z)$ and $\mathcal{A}'' = \mathrm{Coh}(U)$. We obtain a recollement:

$$\mathcal{A}' \begin{smallmatrix} \xleftarrow{\; i^! \;} \\ \xrightarrow{\; i_* \;} \\ \xleftarrow{\; i^* \;} \end{smallmatrix} \mathcal{A} \begin{smallmatrix} \xleftarrow{\; j_* \;} \\ \xrightarrow{\; j^* \;} \\ \xleftarrow{\; j_! \;} \end{smallmatrix} \mathcal{A}''$$

Here i_* is the pushforward of coherent sheaves from Z to X, i^* is the pullback (restriction) to Z, and $i^!$ is given by $\mathrm{Ext}^\bullet(\mathcal{O}_Z, -)$ (shifted appropriately). Similarly, j^* is the restriction to U, while $j_!$ is extension by zero and j_* is the pushforward from U to X. The short exact sequences in the recollement axioms reflect the fact that any coherent sheaf on X sits in exact sequences relating its restrictions to U and Z. This recollement is fundamental to the theory of stratifications by support and provides the geometric foundation for the examples we will encounter in subsequent sections.

Having understood how a category can be decomposed into a subcategory and a quotient via recollement, we are ready to describe how a sequence of such decompositions leads to the notion of stratification.

4.5. Stratifications

While recollements describe how an abelian category can be built from a subcategory and its quotient, stratifications organize a sequence of such recollements along a poset. Thus, a stratification provides a systematic way to decompose an abelian category into simpler layered pieces with these layers fitting together in a coherent way. Such layered structures arise naturally in algebraic geometry and representation theory. In this section, we introduce the formal definition of a stratification and illustrate this concept with a few examples.

Definition 4.5.1 (stratification)**.** Given an abelian category $\mathcal{A}$ and a poset P, a *stratification* of $\mathcal{A}$ with respect to P consists of a family of Serre subcategories, $\mathcal{A}_Q \subseteq \mathcal{A}$ for each subset $Q \subseteq P$, such that the following conditions hold:

(i) $\mathcal{A}_\emptyset = 0$,

(ii) $\mathcal{A}_P = \mathcal{A}$,

(iii) if $Q \subseteq Q'$ are two subsets of P, then $\mathcal{A}_Q \subseteq \mathcal{A}_{Q'}$,

(iv) for every subset $Q \subseteq P$, if we denote by q the maximal element of Q, then there exists a recollement $\mathcal{A}_{Q\setminus\{q\}} \to \mathcal{A}_Q \to \mathcal{A}_{\{q\}}$.

Intuitively, each stratum $\mathcal{A}_q$ represents one layer of the category, and the recollement condition ensures that $\mathcal{A}$ can be built step by step by gluing layers. To build intuition for this definition, we begin with the most elementary examples and gradually increase complexity. The first example shows that the notion of stratification is non-trivial even in the simplest case.

Example 4.5.2. Let $\mathcal{A}$ be any abelian category and let $P = \{0\}$ be the one-element poset. Then there is a unique stratification of $\mathcal{A}$ with respect to P, given by $\mathcal{A}_\emptyset = 0$ and $\mathcal{A}_P = \mathcal{A}$. This example is trivial but important, as it shows that every abelian category admits at least one stratification.

The trivial stratification provides no decomposition whatsoever. While it illustrates the basic formalism, stratifications become most interesting and useful when they arise naturally. We now present an example that appears throughout algebraic geometry.

Example 4.5.3. Let X be a Noetherian scheme and let $\mathcal{A} = \mathrm{Coh}(X)$ be the category of coherent sheaves on X. Suppose X has a finite stratification by locally closed subsets:

$$X = X_0 \sqcup X_1 \sqcup \cdots \sqcup X_n,$$

where each X_i is locally closed and the closures satisfy $\overline{X_0} \subseteq \overline{X_1} \subseteq \cdots \subseteq \overline{X_n}$. Define $P = \{0, 1, \ldots, n\}$ with the usual ordering, and for each $i \in P$, let

$$\mathcal{A}_{\leq i} = \left\{ \mathcal{F} \in \mathrm{Coh}(X) \mid \mathrm{Supp}(\mathcal{F}) \subseteq \overline{X_0} \cup \overline{X_1} \cup \cdots \cup \overline{X_i} \right\}.$$

Each $\mathcal{A}_{\leq i}$ is a Serre subcategory, and these define a stratification of $\mathrm{Coh}(X)$. The stratum $\mathcal{A}_i$ consists of sheaves supported on the closure of X_i but not on any smaller stratum.

To close this part of the notes, we will consider stratifications arising in representation theory, more specifically, in certain categories of modules for Lie superalgebras. We will begin by reviewing the context in which these stratifications arise and then proceed to construct them.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra over $\mathbb{C}$ and let A be an associative, commutative algebra over $\mathbb{C}$. The tensor product $\mathfrak{g} \otimes_{\mathbb{C}} A$ is a super vector space over $\mathbb{C}$, when endowed with the $\mathbb{Z}_2$-grading given by

$$(\mathfrak{g} \otimes_{\mathbb{C}} A)_{\overline{0}} := \mathfrak{g}_{\overline{0}} \otimes_{\mathbb{C}} A \quad \text{and} \quad (\mathfrak{g} \otimes_{\mathbb{C}} A)_{\overline{1}} := \mathfrak{g}_{\overline{1}} \otimes_{\mathbb{C}} A,$$

and admits a unique Lie super bracket that satisfies

$$[x \otimes a,\, y \otimes b]_{\mathfrak{g} \otimes_{\mathbb{C}} A} = [x, y]_{\mathfrak{g}} \otimes (a \cdot_A b)$$

for all $x, y \in \mathfrak{g}$ and $a, b \in A$. Lie superalgebras of this form are known as *map superalgebras*, *generalized current superalgebras*, or *generalized loop superalgebras.*

Since the Lie superalgebra $\mathfrak{g}$ is finite-dimensional and simple, we can choose a Cartan subalgebra $\mathfrak{h}$. The isomorphism classes of finite-dimensional simple $\mathfrak{g}$-modules are parametrized by a subset of $\mathfrak{h}^*$, which is also a subset of the so-called weight lattice of $\mathfrak{g}$ with respect to $\mathfrak{h}$. This weight lattice contains all possible weights of finite-dimensional $\mathfrak{g}$-modules (see [Kac78, Proposition 2.5]).

For every finite-dimensional simple Lie superalgebra $\mathfrak{g}$, there exists a reductive Lie algebra $\mathfrak{r} \subseteq \mathfrak{g}$ for which $\mathfrak{h}_{\overline{0}}$ is the (non-super) Cartan subalgebra (see [BCM19, §2] for more details). We define the category $\mathcal{C}_A$ as the full subcategory of the category of $\mathfrak{g} \otimes A$-modules whose objects are those modules that are equal to the direct sum of their finite-dimensional irreducible $\mathfrak{r}$-submodules. In particular, every object of $\mathcal{C}_A$ is a weight module for $\mathfrak{h}$. The fact that $\mathcal{C}_A$ is an abelian category follows from the fact that the category of $\mathfrak{g} \otimes_{\mathbb{C}} A$-modules is abelian and the observation that the subcategory $\mathcal{C}_A$ is closed under taking kernels, cokernels and finite direct sums.

In the remainder of this section, we will construct a stratification of the abelian category $\mathcal{C}_A$. We begin by constructing Serre subcategories $\mathcal{A}_{\leq \lambda}$ and $\mathcal{A}_{<\lambda}$. For $\lambda \in \mathfrak{h}^*$, denote by $\mathcal{A}_{\leq \lambda}$ the full subcategory of $\mathcal{C}_A$ consisting of $\mathfrak{g} \otimes_{\mathbb{C}} A$-modules M such that the weight space M_μ is non-zero only if $\mu \leq \lambda$.

Similarly, let $\mathcal{A}_{<\lambda}$ be the full subcategory of $\mathcal{A}_{\leq\lambda}$ consisting of modules M such that the weight space M_μ is non-zero only if $\mu < \lambda$. The fact that $\mathcal{A}_{\leq\lambda}$ and $\mathcal{A}_{<\lambda}$ are Serre subcategories of $\mathcal{C}_A$ follows from the observation that they are closed under taking kernels, cokernels, and direct sums.

Now, we will prove that the inclusion functor from every subcategory $\mathcal{A}_{<\lambda}$ to $\mathcal{A}_{\leq\lambda}$ is fully faithful.

Proposition 4.5.4. The inclusion functor $i_* : \mathcal{A}_{<\lambda} \to \mathcal{A}_{\leq\lambda}$ is fully faithful.

Proof. Since $\mathcal{A}_{<\lambda}$ is a full subcategory of $\mathcal{A}_{\leq\lambda}$, we have that

$$\mathrm{Hom}_{\mathcal{A}_{<\lambda}}(N_1, N_2) = \mathrm{Hom}_{\mathcal{A}_{\leq\lambda}}(i_*(N_1), i_*(N_2)) \quad \text{for every } N_1, N_2 \in \mathcal{A}_{<\lambda}. \qquad \square$$

Now, we construct a left adjoint functor to this inclusion functor. That is, we define a functor $i^* : \mathcal{A}_{\leq\lambda} \to \mathcal{A}_{<\lambda}$ for which there exist natural isomorphisms

$$\mathrm{Hom}_{\mathcal{A}_{\leq\lambda}}(M, i_*(N)) \to \mathrm{Hom}_{\mathcal{A}_{<\lambda}}(i^*(M), N).$$

To do that, we assign to each object M of $\mathcal{A}_{\leq\lambda}$, the quotient

$$i^*(M) = M/M_{\not<\lambda}, \quad \text{where} \quad M_{\not<\lambda} := \sum_{\mu\not<\lambda} \mathrm{U}(\mathfrak{g} \otimes A)M_\mu,$$

and assign to each morphism $f \in \mathrm{Hom}_{\mathcal{A}_{\leq\lambda}}(M, M')$, the unique $\mathfrak{g} \otimes_{\mathbb{C}} A$-module homomorphism $i^*(f) : i^*(M) \to i^*(M')$ such that $i^*(f) \circ \pi_M = \pi_{M'} \circ f$, where $\pi_M : M \to i^*(M)$ and $\pi_{M'} : M' \to i^*(M')$ denote the respective projections. Verifying that i^* satisfies the functor axioms is straightforward.

In the following proposition, we prove that this functor i^* is indeed left adjoint to the inclusion functor i_*.

Proposition 4.5.5. The functor $i^* : \mathcal{A}_{\leq\lambda} \to \mathcal{A}_{<\lambda}$ is left adjoint to the inclusion functor $i_* : \mathcal{A}_{<\lambda} \to \mathcal{A}_{\leq\lambda}$.

Proof. To prove that the functor i^* is left adjoint to the functor i_*, we will construct natural isomorphisms

$$\Phi_{M,N} : \mathrm{Hom}_{\mathcal{A}_{\leq\lambda}}(M, i_*(N)) \to \mathrm{Hom}_{\mathcal{A}_{<\lambda}}(i^*(M), N)$$

for all $M \in \mathcal{A}_{\leq\lambda}$ and $N \in \mathcal{A}_{<\lambda}$. To do that, begin by noticing that $i_*(N)$ is simply N viewed as an object of $\mathcal{A}_{\leq\lambda}$. Then, denote by π the canonical projection $M \to i^*(M)$. The natural isomorphism $\Phi_{M,N}$ on a morphism f in $\mathrm{Hom}_{\mathcal{A}_{\leq\lambda}}(M, i_*(N))$ is defined to be the unique homomorphism of $\mathfrak{g} \otimes_{\mathbb{C}} A$-modules $\overline{f} : i^*(M) \to N$ such that $f = \overline{f} \circ \pi$. $\square$

We now construct a right adjoint functor $i^! : \mathcal{A}_{\leq\lambda} \to \mathcal{A}_{<\lambda}$ to the inclusion functor $i_* : \mathcal{A}_{<\lambda} \to \mathcal{A}_{\leq\lambda}$. On objects, we define $i^!(M)$ to be the sum of all $\mathfrak{g} \otimes_{\mathbb{C}} A$-submodules of M that are in $\mathcal{A}_{<\lambda}$. This sum is a $\mathfrak{g} \otimes_{\mathbb{C}} A$-submodule of M contained in $\mathcal{A}_{<\lambda}$ (by construction), as well as its unique largest submodule in $\mathcal{A}_{<\lambda}$. To define $i^!$ on morphisms, let f be a morphism in $\operatorname{Hom}_{\mathcal{A}_{\leq\lambda}}(M, M')$. We define $i^!(f)$ by restricting f, that is, by setting $i^!(f)(m) = f(m)$ for all $m \in i^!(M)$. To verify that this is a well-defined functor is straightforward.

In the next result, we prove that this functor $i^!$ is indeed right adjoint to the inclusion functor.

Proposition 4.5.6. The functor $i^! : \mathcal{A}_{\leq\lambda} \to \mathcal{A}_{<\lambda}$ is right adjoint to the inclusion functor $i_* : \mathcal{A}_{<\lambda} \to \mathcal{A}_{\leq\lambda}$.

Proof. To prove that the functor $i^!$ is right adjoint to the functor i_*, we will construct natural isomorphisms

$$\Phi_{N,M} : \operatorname{Hom}_{\mathcal{A}_{\leq\lambda}}(i_*(N), M) \to \operatorname{Hom}_{\mathcal{A}_{<\lambda}}(N, i^!(M))$$

for all $N \in \mathcal{A}_{<\lambda}$ and $M \in \mathcal{A}_{\leq\lambda}$. Recall that $i_*(N)$ is simply N viewed as an object of $\mathcal{A}_{\leq\lambda}$, and let $j : i^!(M) \hookrightarrow M$ denote the inclusion of the largest submodule of M contained in $\mathcal{A}_{<\lambda}$. Thus, the natural isomorphism $\Phi_{N,M}$ on a morphism $f \in \operatorname{Hom}_{\mathcal{A}_{\leq\lambda}}(i_*(N), M)$ is defined to be the unique homomorphism $\overline{f} : N \to i^!(M)$ such that $f = j \circ \overline{f}$. ◻

The constructions above yield the left half of the stratification diagram

$$\mathcal{A}_{<\lambda} \;\underset{i^*}{\overset{i^!}{\leftleftarrows}}\!\!\xrightarrow{\;i_*\;} \mathcal{A}_{\leq\lambda} \;\underset{j_!}{\overset{j_*}{\leftleftarrows}}\!\!\xrightarrow{\;j^*\;} \mathcal{A}_{\lambda},$$

We now proceed to construct its right half. Begin by defining $\mathcal{A}_\lambda$ to be the full subcategory of $\mathfrak{h} \otimes_{\mathbb{C}} A$-modules consisting of modules N such that

$$(h \otimes 1)n = \lambda(h)n \quad \text{for all } h \in \mathfrak{h} \text{ and } n \in N.$$

In other words, $\mathcal{A}_\lambda$ is the category of $\mathfrak{h} \otimes_{\mathbb{C}} A$-modules on which $\mathfrak{h} \otimes 1$ acts via the weight λ. The fact that the category $\mathcal{A}_\lambda$ is abelian follows from the fact that the category of $\mathfrak{h} \otimes_{\mathbb{C}} A$-modules is abelian and the observation that subcategory $\mathcal{A}_\lambda$ is closed under taking kernels, cokernels, and finite direct sums.

We now define a functor $j^* : \mathcal{A}_{\leq\lambda} \to \mathcal{A}_\lambda$ by assigning: to a module V in $\mathcal{A}_{\leq\lambda}$, its λ-weight space, $j^*(V) := V_\lambda$, and to a morphism $f : M \to N$ in $\mathcal{A}_{\leq\lambda}$, its restriction, $j^*(f) := f|_{M_\lambda} : M_\lambda \to N_\lambda$. To verify that j^* indeed defines a functor is also straightforward.

One can directly prove that j^* is an exact functor. We will obtain this result as a consequence of the construction of left and right adjoint functors for it. To construct these adjoint functors, let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a triangular decomposition of the Lie superalgebra $\mathfrak{g}$. If we let $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ be the corresponding Borel subalgebra of $\mathfrak{g}$, then $\mathfrak{b}^+ \otimes_{\mathbb{C}} A$ is a parabolic subalgebra of $\mathfrak{g} \otimes_{\mathbb{C}} A$.

We will construct a left adjoint functor $j_!$ to j^* using induction from this parabolic subalgebra to the full superalgebra $\mathfrak{g} \otimes_{\mathbb{C}} A$. In this context, the functor $j_! : \mathcal{A}_\lambda \to \mathcal{A}_{\le\lambda}$ is commonly known as the *Weyl functor*.

Define the functor $j_!$ on objects in the following way. To an object N in $\mathcal{A}_\lambda$, we assign $j_!(N) := \mathrm{U}(\mathfrak{g} \otimes A) \otimes_{\mathrm{U}(\mathfrak{b}^+ \otimes A)} N$, where the $\mathfrak{b}^+ \otimes A$-module structure on N is given by extending its $\mathfrak{h} \otimes A$-action trivially on $\mathfrak{n}^+ \otimes A$, the left $\mathrm{U}(\mathfrak{g} \otimes A)$-module structure on $\mathrm{U}(\mathfrak{g} \otimes A)$ is given by left multiplication, and the right $\mathrm{U}(\mathfrak{b}^+ \otimes A)$-module structure on $\mathrm{U}(\mathfrak{g} \otimes A)$ is given by right multiplication. To a homomorphism of $\mathfrak{h} \otimes A$-modules $f : N \to N'$, we assign the tensor morphism $j_!(f) := \mathrm{id}_{\mathrm{U}(\mathfrak{g} \otimes A)} \otimes f$. The fact that $j_!$ is indeed a functor follows from the functorial properties of the tensor product over $\mathrm{U}(\mathfrak{b}^+ \otimes A)$.

In the next result, we prove that $j_!$ is in fact a left adjoint functor to the functor j^*.

Proposition 4.5.7. The functor $j_! : \mathcal{A}_\lambda \to \mathcal{A}_{\le\lambda}$ is left adjoint to the functor $j^* : \mathcal{A}_{\le\lambda} \to \mathcal{A}_\lambda$.

Proof. We will construct natural isomorphisms

$$\Phi_{N,M} : \mathrm{Hom}_{\mathcal{A}_{\le\lambda}}(j_!(N), M) \to \mathrm{Hom}_{\mathcal{A}_\lambda}(N, j^*(M))$$

for all $N \in \mathcal{A}_\lambda$ and $M \in \mathcal{A}_{\le\lambda}$. For a morphism $f \in \mathrm{Hom}_{\mathcal{A}_{\le\lambda}}(j_!(N), M)$, we define the morphism $\Phi_{N,M}(f) \in \mathrm{Hom}_{\mathcal{A}_\lambda}(N, j^*(M))$ by setting

$$\Phi_{N,M}(f)(n) = f(1 \otimes n) \quad \text{for all } n \in N.$$

The fact that $\Phi_{N,M}$ is a natural isomorphism follows from the tensor-Hom adjunction

$$\mathrm{Hom}_{\mathrm{U}(\mathfrak{g} \otimes_{\mathbb{C}} A)}\left(\mathrm{U}(\mathfrak{g} \otimes_{\mathbb{C}} A) \otimes_{\mathrm{U}(\mathfrak{b}^+ \otimes_{\mathbb{C}} A)} N,\, M\right) \cong \mathrm{Hom}_{\mathrm{U}(\mathfrak{b}^+ \otimes_{\mathbb{C}} A)}(N, M). \qquad \square$$

Having established the adjunction between $j_!$ and j^*, we now proceed to show that the functor $j_!$ is fully faithful. To do that, we begin by recalling that a left adjoint functor is fully faithful if and only if the unit of the adjunction is a natural isomorphism. We will use this criterion to prove that the left adjoint functor $j_!$ to j^* is fully faithful.

Proposition 4.5.8. The functor $j_! : \mathcal{A}_\lambda \to \mathcal{A}_{\leq\lambda}$ is fully faithful.

Proof. We will show that the unit of the adjunction between $j_!$ and j^* is a natural isomorphism. Since $\eta : \mathrm{id}_{\mathcal{A}_\lambda} \to j^* \circ j_!$ is a natural transformation, this is equivalent to showing that $\eta_N : N \to j^*(j_!(N))$ is a bijection for every object N in $\mathcal{A}_\lambda$. From the proof of Proposition 4.5.7, we recall that η_N is given by $\eta_N(n) = 1 \otimes n$, where $1 \otimes n$ is viewed as an element of $j_!(N)_\lambda$. The fact that η_N is a bijection follows from the Poincaré-Birkhoff-Witt Theorem (see, for instance, [Hum72, Theorem 17.3]). □

In the stratification of the category $\mathcal{C}_A$, the right adjoint to the functor j^* will be given by a restricted form of coinduction. To explicitly construct this functor, let $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$ denote the opposite Borel subalgebra. Then, assign to an object N in $\mathcal{A}_\lambda$, the *restricted coinduction*

$$j_*(N) := \bigoplus_{\mu \in \mathfrak{h}^*} \mathrm{Hom}_{\mathrm{U}(\mathfrak{b}^- \otimes_{\mathbb{C}} A)} \left(\mathrm{U}(\mathfrak{g} \otimes_{\mathbb{C}} A),\, N\right)_\mu ,$$

where the left $\mathrm{U}(\mathfrak{b}^- \otimes A)$-module structure on N is given by trivially extending its $\mathrm{U}(\mathfrak{h} \otimes A)$-module structure, the left $\mathrm{U}(\mathfrak{b}^- \otimes A)$-module structure on $\mathrm{U}(\mathfrak{g} \otimes A)$ is induced by left multiplication, and the right $\mathrm{U}(\mathfrak{g} \otimes A)$-module structure on $\mathrm{U}(\mathfrak{g} \otimes A)$ is induced by right multiplication. To a morphism $f \in \mathrm{Hom}_{\mathcal{A}_\lambda}(N, N')$, we assign the morphism $j_*(f) : j_*(N) \to j_*(N')$ defined by post-composition, $j_*(f) = - \circ f$. It is straightforward to verify that j_* preserves identities and composition, thus defining a functor. We now establish the right adjointness of j_* with respect to the functor j^*.

Proposition 4.5.9. The functor $j_* : \mathcal{A}_\lambda \to \mathcal{A}_{\leq\lambda}$ is right adjoint to the functor $j^* : \mathcal{A}_{\leq\lambda} \to \mathcal{A}_\lambda$.

Proof. We will construct natural isomorphisms

$$\Phi_{M,N} : \mathrm{Hom}_{\mathcal{A}_{\leq\lambda}}(M, j_*(N)) \to \mathrm{Hom}_{\mathcal{A}_\lambda}(j^*(M), N)$$

for every $M \in \mathcal{A}_{\leq\lambda}$ and $N \in \mathcal{A}_\lambda$. This natural isomorphism is defined on a morphism $f \in \mathrm{Hom}_{\mathcal{A}_{\leq\lambda}}(M, j_*(N))$ to be the morphism $\Phi_{M,N}(f) : M_\lambda \to N$ given by

$$\Phi_{M,N}(f)(m) = f(m)(1) \quad \text{for all } m \in M_\lambda,$$

where $1 \in \mathrm{U}(\mathfrak{g} \otimes_{\mathbb{C}} A)$ denotes the unit element. The fact that $\Phi_{M,N}$ defines a natural isomorphism follows from the tensor-Hom adjunction

$$\mathrm{Hom}_{\mathrm{U}(\mathfrak{g}\otimes_{\mathbb{C}} A)}\left(M,\, \mathrm{Hom}_{\mathrm{U}(\mathfrak{b}^-\otimes_{\mathbb{C}} A)}(\mathrm{U}(\mathfrak{g} \otimes_{\mathbb{C}} A), N)\right) \cong \mathrm{Hom}_{\mathrm{U}(\mathfrak{b}^-\otimes_{\mathbb{C}} A)}(M, N). \qquad \square$$

Having established the adjunction between j^* and j_*, we now proceed to show that the restricted costandard functor j_* is fully faithful. To do that, we begin by recalling that a right adjoint functor is fully faithful if and only if the counit of its corresponding adjunction is a natural isomorphism. We will use this criterion to prove that j_* is fully faithful.

Proposition 4.5.10. The functor $j_* : \mathcal{A}_\lambda \to \mathcal{A}_{\le\lambda}$ is fully faithful.

Proof. We will prove that the counit of the adjunction between j^* and j_* is a natural isomorphism. Since this counit $\varepsilon : j^* \circ j_* \to \mathrm{id}_{\mathcal{A}_\lambda}$ is a natural transformation (by construction), this is equivalent to proving that ε_N is a bijection for every object N in $\mathcal{A}_\lambda$. From the proof of Proposition 4.5.9, we recall that ε_N is explicitly given by $\varepsilon_N(f) = f(1)$ for $f \in j^*(j_*(N)) = j_*(N)_\lambda$. The fact that ε_N is a bijection follows from the weight module structure of $j_*(N)$:

$$j_*(N)_\lambda = \mathrm{Hom}_{\mathrm{U}(\mathfrak{b}^-\otimes_{\mathbb{C}} A)}(\mathrm{U}(\mathfrak{g} \otimes_{\mathbb{C}} A), N)_\lambda \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, N) \cong N. \qquad \square$$

We have thus constructed both halves of the stratification diagram

$$\mathcal{A}_{<\lambda} \underset{i^*}{\overset{i^!}{\leftleftarrows}} \!\!-\, i_* \to \mathcal{A}_{\le\lambda} \underset{j_!}{\overset{j_*}{\leftleftarrows}} \!\!-\, j^* \to \mathcal{A}_\lambda,$$

To conclude this part we will verify that the functors satisfy the conditions (iii) and (iv) of Definition 4.4.1.

The vanishing condition in Definition 4.4.1(iii) follows immediately from the definitions of the functors i_* and j^*. In fact, if M is an object of $\mathcal{C}_A$ for which the weight space M_μ is non-zero only if $\mu < \lambda$, then $j^*(i_*(M)) = M_\lambda = \{0\}$.

To verify the existence of the first exact sequence in Definition 4.4.1(iv), recall that, for every object M of $\mathcal{A}_{\le\lambda}$, the object $i_*(i^*(M))$ is the quotient $M/M_{\not<\lambda}$ viewed as an object of $\mathcal{A}_{\le\lambda}$. Since M is assumed to be in $\mathcal{A}_{\le\lambda}$, its submodule $M_{\not<\lambda}$ is the one generated by M_λ. Hence, the projection $M \to M/\,\mathrm{U}(\mathfrak{g} \otimes_{\mathbb{C}} A)M_\lambda$ is an epimorphism $M \to i_*(i^*(M))$; whose kernel is given by $\mathrm{U}(\mathfrak{g} \otimes_{\mathbb{C}} A)M_\lambda$. Then, notice that $j_!(j^*(M)) = j_!(M_\lambda) = \mathrm{U}(\mathfrak{g} \otimes_{\mathbb{C}} A) \otimes_{\mathrm{U}(\mathfrak{b}^+\otimes_{\mathbb{C}} A)}$

M_λ. Hence, the left $\mathrm{U}(\mathfrak{g} \otimes_{\mathbb{C}} A)$-module structure on M induces a morphism $j_!(j^*(M)) \to M$, explicitly given by $u \otimes m \mapsto u \cdot m$, whose image is exactly $\mathrm{U}(\mathfrak{g} \otimes_{\mathbb{C}} A)M_\lambda$. This means that there exists a natural exact sequence

$$j_!(j^*(M)) \to M \to i_*(i^*(M)) \to 0.$$

To verify the existence of the second exact sequence in Definition 4.4.1(iv), begin by recalling that, for every object M of $\mathcal{A}_{\leq\lambda}$, the object $i_*(i^!(M))$ is its largest submodule whose weights are $< \lambda$. Hence, the inclusion $i_*(i^!(M)) \to M$ is a monomorphism, whose image is $i_*(i^!(M))$ itself. Then, recall that

$$j_*(j^*(M)) = j_*(M_\lambda) = \bigoplus_{\mu \in \mathfrak{h}^*} \mathrm{Hom}_{\mathrm{U}(\mathfrak{b}^+ \otimes_{\mathbb{C}} A)}(U(\mathfrak{g} \otimes_{\mathbb{C}} A), M_\lambda)_\mu.$$

Thus, we can define a morphism $\phi : M \to j_*(j^*(M))$ by setting $\phi(m)(u) = um$. This is a homomorphism of $\mathfrak{g} \otimes_{\mathbb{C}} A$-modules whose kernel is $i_*(i^*(M))$. To justify this claim, notice that ϕ is the image of id_{M_λ} under the adjunction $\mathrm{Hom}_{\mathcal{A}_\lambda}(j_*(M), j_*(M)) \cong \mathrm{Hom}_{\mathcal{A}_{\leq\lambda}}(M, j_*(j^*(M)))$. This implies that ϕ is a well-defined homomorphism of $\mathfrak{g} \otimes_{\mathbb{C}} A$-modules. The fact that $\ker(\phi) = i_*(i^*(M))$ follows from the fact that ϕ maps every weight vector in M whose weight is not $< \lambda$ to 0.

This completes the proof that the category $\mathcal{C}_A$ admits a stratification by the categories $\mathcal{A}_\lambda$:

$$\mathcal{A}_{<\lambda} \underset{i^*}{\overset{i^!}{\leftrightarrows}} \!\!- i_* \to \mathcal{A}_{\leq\lambda} \underset{j_!}{\overset{j_*}{\leftrightarrows}} \!\!- j^* \to \mathcal{A}_\lambda,$$

Part V

Appendices

Appendix A. Groups

A.1. **Axioms and basic examples of groups.** We begin this section with the abstract definition of a group. Groups are one of the most fundamental structures in mathematics, providing a framework for studying symmetry, transformations, and algebraic equations. The definition of a group captures the essential properties of many familiar mathematical objects, such as numbers, vectors, and permutations.

Definition A.1. A *group* is a non-empty set G equipped with a function $m\colon G \times G \to G$ (i.e., a binary operation) satisfying the following conditions:

(i) m is associative, i.e., $m(m(a, b), c) = m(a, m(b, c))$ for all $a, b, c \in G$.

(ii) There exists $e \in G$ such that $m(e, g) = g = m(g, e)$ for all $g \in G$.

(iii) For each $g \in G$, there exists $\tilde{g} \in G$ such that $m(g, \tilde{g}) = e = m(\tilde{g}, g)$.

The element e is called the *identity element* of G. The element $\tilde{g}$ is called the *inverse of* g. A group (G, m) is said to be *commutative* or *abelian* when m is a commutative binary operation, i.e., when $m(g, h) = m(h, g)$ for all $g, h \in G$.

Now we will present some familiar examples of groups. We begin with the integers equipped with their usual addition. This example is foundational, as it illustrates how the abstract definition of a group applies to a well-known mathematical structure.

Example A.2. Consider the set of integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ equipped with the binary operation $m\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $m(a, b) = a + b$. The pair $(\mathbb{Z}, m)$ is an abelian group. In fact:

(i) For all $a, b, c \in \mathbb{Z}$, we have

$$\begin{aligned} m(m(a,b),c) &= m(a+b,c) \\ &= (a+b)+c \\ &= a+(b+c) \\ &= m(a,b+c) \\ &= m(a,m(b,c)). \end{aligned}$$

(ii) The identity element of $(\mathbb{Z}, m)$ is 0. Indeed, for all $a \in \mathbb{Z}$, we have:

$$m(a,0) = a+0 = a = 0+a = m(0,a).$$

(iii) The inverse of an element $a \in \mathbb{Z}$ is $-a \in \mathbb{Z}$. Indeed, for all $a \in \mathbb{Z}$, we have:

$$m(a,-a) = a+(-a) = 0 = (-a)+a = m(-a,a).$$

(iv) Moreover, for all $a, b \in \mathbb{Z}$, we have $m(a,b) = a+b = b+a = m(b,a)$.

The integers under addition are just one example of an abelian group. Another important class of abelian groups arises from vector spaces, which are central to linear algebra.

Example A.3. Let $\Bbbk$ be a field (for example, $\Bbbk = \mathbb{R}$) and $(V, +, \cdot)$ be a $\Bbbk$-vector space. By the definition of a vector space from Linear Algebra, the set V equipped with the binary operation $+ : V \times V \to V$ is an abelian group. In particular, the sets of rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$, and complex numbers $\mathbb{C}$ are abelian groups when equipped with their usual addition.

While addition is a natural binary operation that forms a group, not all familiar operations satisfy the group axioms. In the next example, we will see why the set of real numbers under multiplication does not form a group.

Example A.4. Consider the set of real numbers, $\mathbb{R}$, and the binary operation $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $m(a,b) = ab$. Observe that $(\mathbb{R}, m)$ is **not** a group. Although m is associative and 1 is its identity element, there is no inverse for 0. Indeed, $m(a,0) = 0$ for all $a \in \mathbb{R}$, so there is no $\tilde{0} \in \mathbb{R}$ such that $m(\tilde{0}, 0) = 1$.

Although the set of real numbers under multiplication is not a group, we can modify this example slightly to obtain a group. By excluding the element that causes issues (in this case, 0), we can construct a group structure. This idea is explored in the next example.

Example A.5. Let $(\Bbbk, +, \cdot)$ be a field (for example, $\Bbbk = \mathbb{R}$) and consider the set $\Bbbk \setminus \{0\}$. From the Definition B.1, it follows that $\Bbbk \setminus \{0\}$ equipped with the binary operation $m : \Bbbk \setminus \{0\} \times \Bbbk \setminus \{0\} \to \Bbbk \setminus \{0\}$ given by $m(a, b) = a \cdot b$ is an abelian group.

Having explored examples of groups arising from numbers and vector spaces, we now turn to the smallest possible group. This example is important because it demonstrates that even the simplest non-empty set can be equipped with a group structure.

Example A.6. The set with a single element $\{e\}$ equipped with the unique binary operation $m\colon \{e\} \times \{e\} \to \{e\}$ (given by $m(e, e) = e$) is a (abelian) group. To verify that $(\{e\}, m)$ is indeed a group, notice that

(i) $m(m(e, e), e) = m(e, e) = m(e, m(e, e))$;
(ii) e is the identity element, since $m(e, e) = e$;
(iii) e is the inverse of e, since $m(e, e) = e$;
(iv) $\{e\}$ is abelian, since $m(e, e) = e = m(e, e)$.

This group is called the *trivial group*.

While the trivial group is the smallest possible group, groups can also be constructed from more complex structures, such as sets of functions. In the next example, we will see how the set of bijections on a set forms a group under composition.

Example A.7. Let X be a non-empty set and G be the set of bijections $f : X \to X$. When this set G is equipped with the composition of functions, it becomes a group. To verify this claim, observe that:

(i) Composition of functions is associative, since
$$\begin{aligned}(f \circ (g \circ h))(x) &= f(g \circ h)(x)) \\ &= f(g(h(x)) \\ &= (f \circ g)(h(x)) \\ &= (f \circ g) \circ h)(x), \qquad \text{for all } x \in X.\end{aligned}$$
(ii) The identity function $\mathrm{id}_X : X \to X$, explicitly given by $\mathrm{id}_X(x) = x$ for all $x \in X$ is the identity element, since
$$(f \circ \mathrm{id}_X)(x) = f(\mathrm{id}_X(x)) = f(x) = \mathrm{id}_X(f(x)) = (\mathrm{id}_X \circ f)(x),$$

for all $x \in X$.

(iii) By definition, every bijection $f : X \to X$ has an inverse function f^{-1}, such that $f \circ f^{-1} = \mathrm{id}_X$ and $f^{-1} \circ f = \mathrm{id}_X$.

This group is called the *symmetric group* on X.

In the following sections, we will denote $m(g, h)$ in a simpler way, by either $g \cdot h$, or $g + h$ (in the abelian cases), or gh.

A.2. **Group homomorphisms.** In this section, we will define group homomorphisms. Intuitively, a homomorphism between two groups is a function that preserves the important structure that these sets have, namely their operation. Homomorphisms are essential in group theory because they allow us to compare groups and study their properties in a structured way.

Definition A.8. Let (G, m_G) and (H, m_H) be two groups. A *group homomorphism* from G to H is a function $f\colon G \to H$ satisfying:

(i) $f(m_G(g_1, g_2)) = m_H(f(g_1), f(g_2))$ for all $g_1, g_2 \in G$,

(ii) $f(e_G) = e_H$.

An *isomorphism of groups* is a group homomorphism that is bijective. We say that the group G is *isomorphic* to the group H when there exists an isomorphism of groups $f\colon G \to H$.

To better understand the abstract definition of group homomorphisms, we will consider some concrete examples. We will start with a familiar example from linear algebra and then explore more specialized cases.

Example A.9. Let $\Bbbk$ be a field (for example, $\Bbbk = \mathbb{R}$), and let $(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ be two $\Bbbk$-vector spaces. By definition, every linear transformation $T\colon V \to W$ is a homomorphism from the group $(V, +_V)$ to the group $(W, +_W)$, since

$$T(v_1 +_V v_2) = T(v_1) +_W T(v_2) \qquad \text{for all } v_1, v_2 \in V.$$

Moreover, every linear isomorphism $T\colon V \to W$ is a group isomorphism, since it is a bijection.

A particular case of the previous example arises when we consider specific groups and functions. In the next example, we will see how the exponential function serves as a group isomorphism between the additive group of real numbers and the multiplicative group of positive real numbers.

Example A.10. Consider the additive group $\mathbb{R}$, the multiplicative group $\mathbb{R}_{>0} = \{\alpha \in \mathbb{R} \mid \alpha > 0\}$, and the function $\exp\colon \mathbb{R} \to \mathbb{R}_{>0}$ given by $\exp(a) = e^a$. To verify that exp is a group isomorphism, notice that:

(i) $\exp(a+b) = e^{a+b} = e^a e^b = \exp(a)\cdot\exp(b)$ for all $a, b \in \mathbb{R}$.

(ii) $\exp(0) = e^0 = 1$.

This shows that exp is a group homomorphism. Moreover, $\ln\colon \mathbb{R}_{>0} \to \mathbb{R}$ is the inverse of exp. Therefore, exp is a bijection, and consequently, a group isomorphism.

While the previous examples involve specific functions, there is always a trivial homomorphism between any two groups. This example illustrates the simplest possible homomorphism, which maps every element of the domain group to the identity element of the codomain group.

Example A.11. Let G and H be two groups. Observe that the constant function $f\colon G \to H$ given by $f(g) = e_H$ for all $g \in G$ is a group homomorphism. Indeed:

(i) $f(g_1 g_2) = e_H = e_H e_H = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$.

(ii) $f(e_G) = e_H$.

This homomorphism is called the *trivial homomorphism*. Observe that this homomorphism is an isomorphism if and only if $G = H = \{e\}$.

To finish this section, we will explore the properties of group homomorphisms under composition and the identity function. These properties allow us to define the set of automorphisms of a group, which itself forms a group.

Example A.12. Let G, H, K be groups, and $f\colon G \to H$, $g\colon H \to K$ be group homomorphisms. To verify that the composition $g \circ f\colon G \to K$ is also a group homomorphism notice that:

(i) For all $g_1, g_2 \in G$,
$$\begin{aligned}(g\circ f)(g_1g_2) &= g(f(g_1g_2))\\ &= g(f(g_1)f(g_2))\\ &= g(f(g_1))g(f(g_2))\\ &= (g\circ f)(g_1)(g\circ f)(g_2).\end{aligned}$$

(ii) $(g\circ f)(e_G) = g(f(e_G)) = g(e_H) = e_K$.

This explains why the composition of group homomorphisms is also a group homomorphism.

Next, we will verify that the identity function $\mathrm{id}_G\colon G \to G$ is a group homomorphism. Indeed:

(i) For all $g_1, g_2 \in G$, we have $\mathrm{id}_G(g_1 g_2) = g_1 g_2 = \mathrm{id}_G(g_1)\,\mathrm{id}_G(g_2)$.

(ii) $\mathrm{id}_G(e_G) = e_G$.

Finally, let $\mathrm{Aut}(G)$ denote the set of all group isomorphisms from G to itself. To verify that $\mathrm{Aut}(G)$ is a group under composition, notice that:

(i) the composition of two automorphisms is an automorphism;

(ii) the identity function id_G is an automorphism;

(iii) every automorphism $f\colon G \to G$ has an inverse $f^{-1}\colon G \to G$, which is also an automorphism;

(iv) the composition of functions is associative.

Thus, $\mathrm{Aut}(G)$ is a group, called the *automorphism group* of G.

A.3. **Subgroups.** Intuitively, subgroups are subsets of a group that inherit the group structure of the group in which they are contained. We begin this section with the formal definition of subgroups.

Definition A.13. Let (G, m_G) be a group. A *subgroup* of G is a non-empty subset $H \subseteq G$ satisfying:

(i) If $h_1, h_2 \in H$, then $m_G(h_1, h_2) \in H$.

(ii) If $h \in H$, then $h^{-1} \in H$.

Now, we will present some examples of subgroups. The first example shows that every group has at least one subgroup.

Example A.14. Given any group G, the subsets $\{e_G\}$ and G are subgroups of G. Indeed:

- for $\{e_G\}$, observe that
$$e_G e_G = e_G \in \{e_G\} \quad \text{and} \quad e_G^{-1} = e_G \in \{e_G\}.$$
- for G, observe that, if $g, h \in G$, then
$$gh \in G \quad \text{and} \quad g^{-1} \in G.$$

Thus, $\{e_G\}$ and G are subgroups of G.

Having seen the subgroups that every group contains, we now consider a more interesting example. The integers form a subgroup of the rational numbers under addition, as shown in the next example.

Example A.15. Consider the additive group $(\mathbb{Q}, +)$. Let us show that $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$. Indeed, observe that if $a, b \in \mathbb{Z}$, then $a + b \in \mathbb{Z}$ and $-a \in \mathbb{Z}$.

Subgroups also arise naturally in the context of vector spaces: every vector subspace is a subgroup of the additive group of the vector space. However, not every subgroup of a vector space is a vector subspace, as shown in the next example.

Example A.16. Given a vector space $(V, +, \cdot)$, by definition, every vector subspace is a subgroup of V. (In particular, the subspaces $\{0\}$ and V are subgroups of V – compare with Example A.14.) However, not every subgroup of $(V, +)$ is necessarily a vector subspace of V. For example, we can verify that $\mathbb{Q}$ is a subgroup of the additive group $(\mathbb{R}, +)$. Indeed, observe that if $a, b \in \mathbb{Q}$, then $a + b \in \mathbb{Q}$ and $-a \in \mathbb{Q}$. However, $\mathbb{Q}$ is not a vector subspace of $\mathbb{R}$; for instance, $\pi \in \mathbb{R}$, $1 \in \mathbb{Q}$, but $\pi \cdot 1 = \pi \notin \mathbb{Q}$.

We close this section showing that not all subsets of a group are subgroups. In the next example, we will see that the multiplicative group of non-zero real numbers is not a subgroup of the additive group of real numbers.

Example A.17. The multiplicative group $(\mathbb{R} \setminus \{0\}, \cdot)$ is *not* a subgroup of the additive group $(\mathbb{R}, +)$. Indeed, by Definition A.13, a subgroup is a subset closed under the same operation as the group. In this case, $\mathbb{R} \setminus \{0\}$ is not closed under addition, which is the operation of the group $\mathbb{R}$. Indeed, for every $a \in \mathbb{R} \setminus \{0\}$, we have $a - a = 0 \notin \mathbb{R} \setminus \{0\}$.

A.4. **Kernels and images of homomorphisms.** In this section, we will define kernels and images of group homomorphisms. In addition to providing tools for analysing groups and homomorphisms, these objects also provide tools for constructing subgroups.

Definition A.18. Let $f\colon G \to H$ be a group homomorphism. Define the *kernel* of f as the set

$$\ker(f) = \{g \in G \mid f(g) = e_H\},$$

and define the *image* of f as the image of the function f, that is, the set

$$\operatorname{im}(f) = \{h \in H \mid \text{there exists } g \in G \text{ such that } f(g) = h\}.$$

We turn now to some concrete examples. We will start with a familiar example from linear algebra and then explore more specialized cases.

Example A.19. Consider two $\mathbb{R}$-vector spaces V and W, viewed as groups. Recall from Example A.9 that every linear transformation $T\colon V \to W$ is a group homomorphism. Moreover, the kernel of T as a linear transformation is the same as the kernel of T as a group homomorphism, and the image of T as a linear transformation is the same as the image of T as a group homomorphism.

While the previous example involves vector spaces, kernels and images also play an important role in more general group homomorphisms. In the next example, we will see how the trivial homomorphism provides a simple but instructive case.

Example A.20. Consider two groups G, H, and the trivial homomorphism $f : G \to H$, given explicitly by $f(g) = e_H$ for all $g \in G$ (see Example A.11). By construction, $f(g) = e_H$ for all $g \in G$, so $\ker(f) = G$. Additionally, the only element $h \in H$ such that there exists $g \in G$ satisfying $f(g) = h$ is $h = e_H$. This shows that $\operatorname{im}(f) = \{e_H\}$ (the trivial subgroup).

Another important example arises when we consider the canonical projection from the integers to the integers modulo n. This example illustrates how kernels and images can be used to analyse quotient groups.

Example A.21. Let $n > 1$ and recall that the set $\mathbb{Z}_n := \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ is a group under the addition modulo n. Then, consider the canonical projection $f : \mathbb{Z} \to \mathbb{Z}_n$, given explicitly by $f_1(z) = \overline{z}$.

The kernel of f is $\{kn \mid k \in \mathbb{Z}\}$. Indeed, on one hand, if $z \in \{kn \mid k \in \mathbb{Z}\}$, then $f(z) = f(kn) = \overline{kn} = \overline{0} \in \mathbb{Z}_n$. On the other hand, if $\overline{z} = \overline{0} \in \mathbb{Z}_n$, this means that the remainder of the division of z by n is 0; that is, n divides z. Therefore, $\ker(f) = \{kn \mid k \in \mathbb{Z}\}$.

Now, we will verify that the image of f is $\mathbb{Z}_n$. Indeed, for each $h \in \mathbb{Z}_n$, we can choose an element $g \in \{0, 1, \ldots, n-1\} \subseteq \mathbb{Z}$ to obtain $f(g) = h$. This shows that f is surjective

The next result shows that kernels and images of homomorphisms are always subgroups of the domain and codomain respectively of the corresponding homomorphism.

Proposition A.22. If $f\colon G \to H$ is a group homomorphism, then $\ker(f)$ is a subgroup of G and $\operatorname{im}(f)$ is a subgroup of H.

Proof. See [DF04, §3.1, Proposition 1]. □

A.5. **Quotient groups and normal subgroups.** The main goal of this section is to define quotient groups, that is, quotients of a group by one of its subgroups. For this quotient to also have a group structure, the subgroup by which we quotient must satisfy a certain condition. Subgroups that satisfy this condition are called *normal*. We begin this section by constructing the quotient as a set.

Consider a group G and a subgroup $N \subseteq G$. Define a relation on G as follows:

$$g \sim h \quad \text{if and only if} \quad h^{-1}g \in N.$$

We will verify that $\sim$ is an equivalence relation:

- For every $g \in G$, we have $g \sim g$. Indeed, since N is a subgroup of G, it follows that $g^{-1}g = e_G \in N$.
- If $g \sim h$, then $h^{-1}g \in N$ by definition. Since N is a subgroup of G, it follows that $g^{-1}h = (h^{-1}g)^{-1} \in N$. Hence, $h \sim g$.
- If $a \sim b$ and $b \sim c$, then $b^{-1}a, c^{-1}b \in N$. Since K is a subgroup of G, it follows that $c^{-1}a = (c^{-1}b)(b^{-1}a) \in N$. Hence, $a \sim c$.

Denote by G/N the set of equivalence classes of the relation $\sim$, denote by $\bar{g} \in G/N$ the equivalence class represented by the element $g \in G$, and by gN (resp. Ng) the subset $\{gn \in G \mid n \in N\}$ (resp. $\{ng \in G \mid n \in N\}$). A set of the form gN (resp. Ng) is called a *left coset of g* (resp. *right coset of g*). Observe that $\bar{h} = \bar{g}$ if and only if $h \in gN$. That is, the elements of the left coset of g are the representatives (in G) of the equivalence class $\bar{g}$ (which is an element of G/N).

Definition A.23. Given a group G, a subgroup $N \subseteq G$ is said to be *normal* if $gN = Ng$ for every $g \in G$.

The next result shows that a necessary and sufficient condition for the quotient G/N to admit a group structure when equipped with the operation induced from the group G is that the subgroup N is normal. This fact highlights the relevance of normal subgroups and justifies their definition.

Proposition A.24. Let G be a group and $N \subseteq G$ a subgroup.

(a) Equipped with the operation $m\colon (G/N) \times (G/N) \to (G/N)$ given by $m(\bar{g}, \bar{h}) = \overline{gh}$, the set G/N is a group if and only if N is a normal subgroup of G.

(b) If N is a normal subgroup of G, then the function $f\colon G \to G/N$ given by $f(g) = \bar{g}$ is a group homomorphism and $\ker(f) = N$.

Proof. See [DF04, §3.1, Proposition 5]. □

By Proposition A.24(a), $N \subseteq G$ is a normal subgroup if and only if G/N is a group. In the following examples, we will construct normal subgroups and their respective quotient subgroups.

Example A.25. For every group G, the trivial subgroup $\{e_G\} \subseteq G$ is a normal subgroup. Indeed, $g\{e_G\} = \{g\} = \{e_G\}g$ for every $g \in G$. Moreover, the quotient group $G/\{e_G\}$ is isomorphic to G. Indeed, consider the function $f : G \to G/\{e_G\}$ given by $f(g) = \bar{g}$. By Proposition A.24(b), f is a group homomorphism whose kernel is $\{e_G\}$. Thus, f is injective. Additionally, since every element of $G/\{e_G\}$ is, by construction, of the form $\bar{g}$ for some $g \in G$, it follows that f is surjective. This shows that f is an isomorphism between G and $G/\{e_G\}$.

Having seen the simplest example of a normal subgroup and its quotient, we now consider the opposite extreme: the case where the subgroup is the entire group. This example illustrates how the quotient group can sometimes be trivial.

Example A.26. For every group G, the subgroup $G \subseteq G$ is normal. Indeed, $gG = \{gh \mid h \in G\} \subseteq G$ and, for a fixed $g \in G$, for any $k \in G$, we have $h = g^{-1}k \in G$ and $gh = g(g^{-1}k) = k$. This shows that $gG = G$. The proof that $Gg = G$ is completely analogous. With this, we conclude that G is a normal subgroup of G.

Moreover, the quotient group G/G is the trivial group. To verify this claim, we will show that G/G contains only one element; that is, we will show that

$\overline{g} = \overline{e_G} \in G/G$ for every $g \in G$. This is true because ${e_G}^{-1}g = g \in G$ for every $g \in G$. Therefore, G/G contains only one element, and hence, it is the trivial group.

Next, we turn to a more interesting example involving the permutation group S_3. This example demonstrates how normal subgroups can be identified using homomorphisms and how they relate to quotient groups.

Example A.27. Let X be a non-empty set and G be the symmetric group defined in Example A.7. When X has 3 or more elements, we can construct a subgroup of G that is not normal.

If X has three or more distinct elements, we can fix three of them, x_1, x_2, x_3. Using the first two elements, we can define a function $f \in G$ in the following way:

$$f(x_1) = x_2, \quad f(x_2) = x_1 \quad \text{and} \quad f(x) = x \text{ for all } x \in X \setminus \{x_1, x_2\}.$$

The subset $H := \{\mathrm{id}_X, f\}$ is a subgroup of G, since

$$\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_x, \quad \mathrm{id}_X \circ f = f \circ \mathrm{id}_X = f, \quad f \circ f = \mathrm{id}_X,$$
$$\mathrm{id}_X^{-1} = \mathrm{id}_X \quad \text{and} \quad f^{-1} = f.$$

However, the subgroup H is not normal in G. To justify this claim, we will construct an element $g \in G$ such that $gH \neq Hg$. Namely, let $g : X \to X$ be the function defined by

$$g(x_2) = x_3, \quad g(x_3) = x_2 \quad \text{and} \quad g(x) = x \text{ for all } x \in X \setminus \{x_2, x_3\}.$$

Notice that the subset gH consists of the functions g and $g \circ f$, while the subset Hg consists of the functions g and $f \circ g$. To justify that $gH \neq Hg$, we have to show that $g \circ f \neq f \circ g$. In fact,

$$(g \circ f)(x_1) = g(x_2) = x_3 \quad \text{and} \quad (f \circ g)(x_1) = f(x_1) = x_2.$$

This shows that $g \circ f \neq f \circ g$, and as a consequence, that $gH \neq Hg$ and that H is not a normal subgroup of G

In abelian groups, the situation is simpler: every subgroup is normal. This is because the group operation is commutative, which ensures that left and right cosets coincide.

Example A.28. Let G be an abelian group and $H \subseteq G$ a subgroup. To verify that H is normal, notice that, for every $g \in G$ and $h \in H$, we have $gh = hg$

because G is abelian. Thus, $gH = \{gh \mid h \in H\} = \{hg \mid h \in H\} = Hg$. This shows that H is a normal subgroup of G.

Finally, we revisit the connection between kernels and normal subgroups. The kernel of any group homomorphism is always a normal subgroup, as shown in the following example.

Example A.29. Let $f : G \to H$ be a group homomorphism. Recall from Proposition A.22 that $\ker(f)$ is a subgroup of G. To show that $\ker(f)$ is normal, notice that

$$\begin{aligned} f(gkg^{-1}) &= f(g)f(k)f(g^{-1}) \\ &= f(g)e_H f(g^{-1}) \\ &= f(g)f(g^{-1}) \\ &= f(gg^{-1}) \\ &= f(e_G) \\ &= e_H. \end{aligned}$$

for all $g \in G$ and $k \in \ker(f)$. This means that $gkg^{-1} \in \ker(f)$, or equivalently, that $g\ker(f) \subseteq \ker(f)g$. One can verify that the other inclusion is also true in a completely analogous way. This implies that $\ker(f)$ is a normal subgroup of G.

A.6. **Isomorphism Theorems.** The theme of this section are the Isomorphism Theorems. These results are very important for group theory and have numerous applications.

The First Isomorphism Theorem is analogous to the Rank-Nullity Theorem in Linear Algebra. It states that, for every group homomorphism, there exists an isomorphism between its image and the quotient of its domain by its kernel.

Theorem A.30. For every group homomorphism $f\colon G \to H$, there exists a group isomorphism $G/\ker(f) \cong \operatorname{im}(f)$.

Proof. See [DF04, §3.3, Theorem 16]. □

The Second Isomorphism Theorem is a result that allows for cancellations in group quotients.

Theorem A.31. Let G be a group and $H, K \subseteq G$ subgroups. If $H \subseteq N_G(K)$, then: HK is a subgroup of G, K is normal in HK, $(H \cap K)$ is normal in H, and there exists a group isomorphism $HK/K \cong H/(H \cap K)$.

Proof. See [DF04, §3.3, Theorem 18]. □

The Third Isomorphism Theorem, like the second, allows for cancellations in quotients. In this case, the cancellation is done with respect to the normal subgroup by which we quotient (the "denominator"). This theorem has a very important consequence: it establishes relationships between the normal subgroups of a group and those of its quotients.

Theorem A.32. Let G be a group. If $H \subseteq K$ are normal subgroups of G, then K/H is a normal subgroup of G/H and there exists a group isomorphism

$$\frac{G/H}{K/H} \cong G/K.$$

Proof. See [DF04, §3.3, Theorem 19]. □

Appendix B. Rings

A ring is a set equipped with two binary operations, typically called addition and multiplication, that satisfy certain axioms. They generalize integers, polynomials, and other familiar mathematical objects. In this section, we will define rings, explore their properties, and provide concrete examples to illustrate these concepts.

Definition B.1. A *ring* R is a set equipped with two binary operations

$$s\colon R \times R \to R \quad \text{and} \quad m\colon R \times R \to R,$$

satisfying the following conditions:

(i) (R, s) is an abelian group,

(ii) $m(a, m(b, c)) = m(m(a, b), c)$ for all $a, b, c \in R$,

(iii) $m(s(a, b), c) = s(m(a, c), m(b, c))$ for all $a, b, c \in R$,

(iv) $m(a, s(b, c)) = s(m(a, b), m(a, c))$ for all $a, b, c \in R$.

The identity element of the group (R, s) will be denoted by 0_R. A ring (R, s, m) is said to be *commutative* if

$$m(a, b) = m(b, a) \qquad \text{for all } a, b \in R.$$

A ring (R, s, m) is said to have an *identity* if there exists $1_R \in R$ such that

$$m(1_R, a) = a = m(a, 1_R) \qquad \text{for all } a \in R.$$

A ring (R, s, m) is called a *division ring* if it has an identity and $(R \setminus \{0_R\}, m)$ is a group (i.e., every non-zero element of R has a multiplicative inverse). A ring (R, s, m) is called a *field* if it is a commutative division ring (in particular, (R, s) and $(R \setminus \{0_R\}, m)$ are abelian groups).

To better understand the abstract definition of rings, we will consider some concrete examples. We will start with the simplest possible ring and then move to more familiar examples like the ring of integers and polynomial rings.

Example B.2. Consider a set with a single element, $\{0\}$, and define the binary operations:

$$s\colon \{0\} \times \{0\} \to \{0\} \quad \text{by} \quad s(0,0) = 0,$$
$$m\colon \{0\} \times \{0\} \to \{0\} \quad \text{by} \quad m(0,0) = 0.$$

We will verify that $(\{0\}, s, m)$ is a ring.

(i) $(\{0\}, s)$ is the trivial group,

(ii) $m(0, m(0,0)) = m(0,0) = 0$ and $m(m(0,0), 0)) = m(0,0) = 0$,

(iii) $m(s(0,0), 0) = m(0,0) = 0$ and $s(m(0,0), m(0,0)) = s(0,0) = 0$,

(iv) $m(0, s(0,0)) = m(0,0) = 0$ and $s(m(0,0), m(0,0)) = m(0,0) = 0$.

Observe that $0_{\{0\}} = 0$. Additionally, $\{0\}$ is a commutative ring with identity $1_{\{0\}} = 0$. Indeed, $m(0,0) = 0 = m(0,0)$. However, $\{0\}$ is not a division ring (and consequently not a field), since $\{0\} \setminus \{0_{\{0\}}\} = \emptyset$ is not a group.

This ring is called the *trivial ring*.

Having seen the simplest example of a ring, we now consider a more familiar example: the ring of integers.

Example B.3. Consider the set $\mathbb{Z}$ (of integers) equipped with the binary operations

$$s\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \quad \text{given by} \quad s(a,b) = a + b,$$
$$m\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \quad \text{given by} \quad m(a,b) = ab.$$

We will verify that $(\mathbb{Z}, s, m)$ is a ring.

(i) We have seen in Example A.2 that $(\mathbb{Z}, s)$ is a group.

(ii) for all $a, b, c \in \mathbb{Z}$,
$$m(a, m(b, c)) = m(a, bc) = a(bc) = (ab)c = m(ab, c) = m(m(a, b), c)).$$
(iii) for all $a, b, c \in \mathbb{Z}$,
$$m(s(a, b), c) = m(a+b, c) = (a+b)c = ac+bc = s(ac, bc) = s(m(a, c), m(b, c)).$$
(iv) for all $a, b, c \in \mathbb{Z}$,
$$m(a, s(b, c)) = m(a, b+c) = a(b+c) = ab+ac = s(ab, ac) = s(m(a, b), m(a, c)).$$

Observe that $0_{\mathbb{Z}} = 0$. Additionally, $\mathbb{Z}$ is a commutative ring with identity $1_{\mathbb{Z}} = 1$. Indeed, $m(a, b) = ab = ba = m(b, a)$ and $m(1, a) = a = m(a, 1)$ for all $a, b \in \mathbb{Z}$. However, $\mathbb{Z}$ is not a division ring (and consequently not a field). Indeed, $m(2, a) = 1$ if and only if $a = \frac{1}{2}$. Since $\frac{1}{2} \notin \mathbb{Z}$, the element $2 \in \mathbb{Z} \setminus \{0\}$ does not have a multiplicative inverse.

To close this appendix, we will show that the set of integers modulo n, is also a ring.

Example B.4. The set of integers modulo n, denoted $\mathbb{Z}_n$, is also a ring. The operations of addition and multiplication are defined as follows:
$$s\colon \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n \quad \text{given by} \quad s(\overline{a}, \overline{b}) = \overline{a+b},$$
$$m\colon \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n \quad \text{given by} \quad m(\overline{a}, \overline{b}) = \overline{ab}.$$
We will verify that $(\mathbb{Z}_n, s, m)$ is a ring.

(i) Recall that $(\mathbb{Z}_n, s)$ is a group.

(ii) for all $a, b, c \in \mathbb{Z}$, we have
$$m(\overline{a}, m(\overline{b}, \overline{c})) = m(\overline{a}, \overline{bc}) = \overline{a}(\overline{bc}) = \overline{abc} = (\overline{ab})\overline{c} = m(\overline{ab}, \overline{c}) = m(m(\overline{a}, \overline{b}), \overline{c})).$$
(iii) for all $a, b, c \in \mathbb{Z}$, we have
$$\begin{aligned} m(s(\overline{a}, \overline{b}), \overline{c}) &= m(\overline{a+b}, \overline{c}) \\ &= (\overline{a+b})\overline{c} \\ &= \overline{ac} + \overline{bc} \\ &= s(\overline{ac}, \overline{bc}) \\ &= s(m(\overline{a}, \overline{c}), m(\overline{b}, \overline{c})). \end{aligned}$$

(iv) for all $a, b, c \in \mathbb{Z}$, we have

$$\begin{aligned} m(\overline{a}, s(\overline{b}, \overline{c})) &= m(\overline{a}, \overline{b+c}) \\ &= \overline{a}(\overline{b+c}) \\ &= \overline{ab} + \overline{ac} \\ &= s(\overline{ab}, \overline{ac}) \\ &= s(m(\overline{a}, \overline{b}), m(\overline{a}, \overline{c})). \end{aligned}$$

Observe that $0_{\mathbb{Z}} = \overline{0}$. Additionally, $\mathbb{Z}_n$ is a commutative ring with identity $1_{\mathbb{Z}} = \overline{1}$. Indeed, $m(\overline{a}, \overline{b}) = \overline{ab} = \overline{ba} = m(\overline{b}, \overline{a})$ and $m(1, \overline{a}) = \overline{a} = m(\overline{a}, 1)$ for all $a, b \in \mathbb{Z}$.

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